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Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics

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Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics

by

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Excellent Teaching: A Collective Case Study of Outstanding

Elementary Mathematics Teachers' Teaching of Mathematics

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This qualitative collective case study explored the mathematical teaching of three excellent elementary teachers who were nominated by experts in mathematics and mathematics educational organizations, agencies and universities. I examined what excellent elementary mathematics teachers know and do in their practice of teaching. The study depicts detailed verbatim interactions between the teachers and students during actual teaching episodes to give the reader naturalistic examples of the explanation patterns and questioning strategies that these excellent teachers used to further students' understandings of mathematical concepts and procedures. Analyses of the pedagogical strategies, including the interactive exploratory problem solving format these teachers used, and explanations for those decisions from the teacher and the researcher are included. The teachers were studied individually and collectively in an exploration of what excellent elementary mathematics teaching is through the lens of excellent teachers' teaching practices in their classrooms and then compared with research literature.

The findings show examples of how these teachers enhanced their students' experiences with mathematics and identified six dimensions of excellent teaching. These teachers enabled students to become active agents of their own learning and showed how to allow students to construct their own understandings without telling them what to think

or do. They empowered students by accepting them as capable thinkers who can reason and provide proof for that reasoning in early schooling experiences. These teachers encouraged students to develop and showcase their own thinking and non-routine algorithm discoveries to their peers showing multiple pathways to problem solutions and how they relate to mathematical ideas. Student Understanding of mathematics was described in examples of classroom episodes. The complexity of the teaching and learning of mathematics is explored finding unique interdependent and interwoven relationships between mathematical concepts and procedures. It was found that these teachers used cognitive approaches to mathematics teaching and taught their students based on what the students currently understood.

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M.J.G

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Chapter 1

Introduction

The mathematics a person needs to know to be productive in the 21st century world is drastically different than just a generation ago. The idea that the only mathematics an everyday person needs is to be able to execute basic computations is outdated and harmful to one's advancement. Today's world is filled with technological advancements that are changing at alarming rates. Our citizens must be able to use mathematical reasoning and logic, and problem solve to adapt to those changes. "People must be prepared to learn, analyze, and use mathematical ideas they have never encountered in school or used before" (Rand, 2003). Children must have increased mathematical proficiency to achieve adequate employment in the current and future workplace. In 1950, 80% of jobs were classified as "unskilled;" now, an estimated 85% of all jobs are classified as "skilled" (Day & McCabe, 1997).

Mathematics Education History in the United States

Concern of student performance and the quality of mathematics education in the United States public school system has a longstanding history. The emphasis of mathematics education has changed, in a cyclical pattern, from a focus on arithmetic to curriculum and back to pedagogy. It is important to discuss briefly some of the history of mathematics education in the United States to get a deeper understanding of the concerns that have developed over the years. These historical experiences and influences can inform us about current controversies as to what is considered excellent mathematics teaching.

The children that attended public school in the early 20th century usually completed the eighth grade. High school was reserved for the elite. Fewer than 7% of the 14-year-olds in the United States were enrolled in high school, with roughly half of those graduating (Stanic, 1987). The level of mathematics training for elementary and high school students was dramatic. Elementary curriculum consisted of basic "arithmetic." The official curriculum for the course of study in the state of Nebraska in the mid-1930's state (Taylor, 1936):

> General Aims in Teaching Arithmetic
>
> 1. To develop such accuracy and speed in arithmetical processes as are required in the usual transactions of life.
> 2. To develop a complete, accurate, and instantaneous knowledge of number combinations involving addition, subtraction, multiplication, and division.
> 3. To develop a ready knowledge of practical measurements, arithmetical terms, and problem procedure.
> 4. To relate arithmetic to the common, familiar affairs of life and business relations.
> 5. To develop habits of neatness, accuracy, speed, logical procedure, perseverance, and self-reliance.

The high school mathematics curriculum included algebra, geometry, and physics and students were held to high standards (Schoenfeld, 2004). In the 1909-1910 school year, roughly 57% of the nation's high school students studied algebra and more than 31% studied geometry and 1.9% studied trigonometry (Jones & Coxford, 1970). Most of the children in the United States received a very basic education in mathematics. In general, roughly 4% of the children studied algebra, roughly 2% studied geometry, and under 1% of America's children studied trigonometry in the early 20th century.

In the 1940's there was an influx of students coming into the public school system in the United States. Almost three fourths of the children aged 14 to 17 attended high school, and 49% of the 17-year-olds graduated (Stanic, 1987). The mathematics curriculum remained the same, but the student body was much more diverse and ill prepared (Schoenfeld, 2004). The percentage of students enrolled in high school mathematics began to drop from 1909 to 1949, from 57% to 27% in the case of algebra and from 31% to 13% in the case of geometry (Klein, 2003).

In 1957, the Soviet Union sparked America's interest and concern for improved mathematical excellence with the launching of Sputnik. The United States became concerned about national security and its status as a world leader. This event occurred during the Cold War and world domination by the Soviets was feared. The nation supported advancements in mathematics curriculum that came to be known as the "new math." The National Science Foundation (NSF) developed a new curriculum in mathematics that was more modern, because for the first time, some of the content really was new: aspects of set theory, modular arithmetic, and symbolic logic were embedded in the curriculum (Schoenfeld, 2004). This new math era had its complications. The top-down nature of curriculum change left stakeholders in uncomfortable positions. Teachers were aware of the new changes, but some of the teachers were not prepared or trained to deliver the new curriculum. Some parents did not know how to help their child and could not understand the value of the new curriculum.

By the early 1970's it was accepted that "New Math was dead" (Klein, 2003). The direction for the United States was to get back to the basics. Basically, the curriculum focused largely on skills and procedures and resembled pre-Sputnik curriculum: arithmetic in the 1st through 8th grades, algebra in the 9th grade, geometry in the 10th grade, a 2nd year of algebra and sometimes trigonometry in the 11th grade, and precalculus in the 12th grade (Schoenfeld, 2004). Results of student achievement in the

1970's showed little ability of students to problem solve. This could have been expected with the focus of the curriculum on procedural computation. Students did not improve their performance on the "basics" either (Schoenfeld, 2004). In response, The National Council of Teachers of Mathematics (NCTM) proposed a problem solving format to replace the back to the basics movement. Problem solving was in its infancy during this period of history in the public schools. For example, a problem solving problem generally looked like the following:

> Mike had ten apples. He gave three apples to LaVeta. How many apples does he have left? The numbers and notations (10 – 3 = 7) were replaced with words in a very basic way.

In the 1980's the crisis moved from military to economic. Some international countries were considered superior to the United States in economical efficiency and the public was also concerned about the rising national deficit. In mathematics, the United States performed poorly on the Second International Mathematics Study (McKnight, Travers, Crosswhite, & Swafford, 1987). The National Commission on Excellence in Education (1983) was appointed to examine the quality of education in the United States. The result was an influential document called "A Nation at Risk." The report began as follows:

> What was unimaginable a generation ago has begun to occur – others are matching and surpassing our educational attainments. If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war. As it stands, we have allowed this to happen to ourselves. We have even squandered the gains in student achievement made in the wake of the Sputnik challenge. Moreover, we have dismantled essential support systems which helped make those gains possible. We have, in effect, been committing an act of unthinking, unilateral educational disarmament. (p. 1)

These developments led the way to a new movement in mathematics education in the United States. The NCTM assembled a new standards-based curriculum called Curriculum and Evaluation Standards for School Mathematics (1989). This "standards movement" was consistent with available research about excellent mathematics teachings and took the nation by storm (Schoenfeld, 2004). Other disciplines and government entities began recommending similar epistemological ideas to a standards-based reform movement that some scholars and government leaders support today.

Student Performance on Standardized Assessments

These reform movements have been instituted in an attempt to improve the performance of student achievement on standardized assessments. Throughout history, results of American children in standardized assessments in mathematics have not reached the level of accomplishment that scholars, parents, colleges, business, and government expect. Such concerns are far from new, over 150 years ago; Horace Mann administered a survey to Boston schoolchildren. He was dismayed that only one-third of the students could answer the arithmetic questions. "Such a result repels comment," he said. "No friendly attempt at palliation can make it any better. No severity of just censure can make it any worse" (p. xii). In 1919, when the survey was administered in school districts around the country, the results for arithmetic were even worse than they had been in 1845 (NRC, 2001). This pattern of student performance has generally not improved throughout the 20th century.

Since the 1970's the United States has relied primarily on two assessment results that measure student performance; The Third International Mathematics and Science Study (TIMSS), and the National Assessment of Educational Progress (NAEP). The

TIMSS results compare scores by countries. These results can indicate where students in the United States compare to students from other countries by the average score of all students. The NAEP results are domestic and measure a student's proficiency. The NAEP measures students in four categories: "Below Basic," "Basic," "Proficient," and "Advanced." This assessment is considered the Nation's Report Card and is used to compare data of states. NAEP data show that there is a significant performance gap between students of different ethnic backgrounds. Even though this study did not examine multicultural aspects of proficiency in mathematics education learning about what excellent teaching is could help teachers narrow the achievement gap and afford success to all students.

Recent results of United States students on both the TIMSS and NAEP "echo a dismal message of lackluster performance, now three decades old" and "unacceptable" (NCMST, 2000). In the 1995 TIMSS results, 7 countries scored higher and 12 countries scored lower than the United States in 4th grade mathematics. This received an "above the international average" performance (NCMST, 2000). In the same assessment, the results for 8th graders, 20 countries scored higher and 7 countries scored lower than the United States receiving a "slightly below international average" rating. In 12th grade, 14 countries scored higher and 2 countries scored lower than the United States. "However, among 20 nations assessed in advanced mathematics, none scored significantly lower than the United States" (NCMST, 2000).

The NAEP results of 1996 scores were also disappointing. Across the grades (4th, 8th, and 12th) roughly 35% of the students were below the basic level of achievement and another 45% or so were at that level, which is defined as denoting "partial mastery of knowledge and skills that are fundamental for proficient work" (NRC, 2001). In the 2000 NAEP results, only 17% of 12th grade students performed above a basic level of performance (NCES, 2001).

Reform Efforts in United States Public Schools

Many reformers in the 20th century have tried to improve mathematical proficiency. During the New Math Era, 1950's through the 1970's, emphasis for improvement was on improving the mathematics curriculum (NRC, 2001). In the last few decades, reformers are also concerned about mathematics teaching (Ball, 2000; Kamii, 1989; Lambert, 2001; Ma, 1999; Stigler & Hiebert, 1999) and assessment through No Child Left Behind legislation and the National Council of Teachers of Mathematics (NCTM) (1989, 2000) Standards and Principles.

Many attempts to improve mathematics education have focused on the teaching of mathematics in the elementary grades (CBMS, 2001; Hill, Schilling, & Ball, 2004; NRC, 2001) as well as the No Child Left Behind legislation, Mathematical Association of America (MAA), National Council of Teachers of Mathematics (NCTM), American Mathematical Society (AMS), National Science Foundation (NSF), and others. For example, the NCTM devised a standards based program to guide schools and teachers about what needs to be taught and in what ways are effective. The National Science Foundation provides funds for educational programs and research to determine how best to teach mathematics and what practices achieve proficiency. Many scholars, mathematicians, and mathematics educators believe that elementary teachers are unprepared to teach mathematics well. It is assumed that future teachers enter college with insufficient knowledge of mathematics, receive little mathematics education, and are unprepared to teach mathematics (Ball, Lubienski, & Mewborn, 2001; CBMS, 2001; Ma, 1999; NCMST, 2000). Professional commissions (RAND Mathematical Study Panel, National Research Council, National Commission on Mathematics and Science Teaching, Conference Board of the Mathematical Sciences) and organizations (National Science Foundation, Mathematical Association of America, Study for Instructional Improvement, among others) have been assembled to analyze and recommend changes that could have positive effects on improving elementary mathematics teaching and student achievement. The recommendations from these groups generally consists of improving teacher content knowledge, quality teacher recruitment, advances in teacher education that consists of mathematical knowledge for teaching (specialized knowledge), constructivist teaching, and professional development opportunities and programs. The results of these efforts have yet to be realized in actual reform teaching practice in the classroom and student achievement for all (Ball & Bass, 2003; Ferrini-Mundy, 1997; Stigler & Hiebert, 1999).

In the last decade, education has been monitored by accountability standards for reading and mathematics. Many stakeholders believe that holding teachers and schools accountable for student performance will improve student and teacher achievement. Frustrated by poor student performance and perceived unwillingness by teachers to improve has prompted lawmakers to enact accountability legislation, called the No Child Left Behind Act, that threatens sanctions for schools if standards are not met. Much research and professional development at the federal, collegiate and school district levels have attempted to assist elementary teachers in mathematical teaching and learning in content knowledge and specialized knowledge for teaching mathematics. Results of these efforts show that classroom practice and student achievement has generally not changed (Ball, Hill, & Bass, 2005; Cohen & Ball, 2000; Stigler & Hiebert, 1999). The failure of these reforms to affect classroom practice has baffled many scholars (Cohen, 1989; McLaughlin, 1990; Sarason, 1971; Skemp, 1989; Tyack & Cuban, 1995). In summary from the findings above, some of the most frequent explanations of past reform failures were: culturally embedded views of knowledge, learning and teaching, social organizations of schools, politics, curriculum and assessment materials, and teacher education programs.

Historically, mathematicians and mathematics educators were separate entities both having their own passions and philosophies about mathematics and neither was necessarily interested in getting involved in the others work. In the last decade or so, mathematicians have taken a more active role in the reform movement through pedagogical and curricular contributions. The teaching of mathematics is quite different from the work of a mathematician. The teacher is concerned about how the students are learning specific mathematics that has already been proven and accepted as true. The mathematician's work involves creating new theorems, conjectures, and knowledge about a piece of mathematics that will undertake scrutiny by other mathematicians before it is considered true. The teacher and mathematician see mathematics differently as well. The teacher sees mathematics as a static subject (content area) that students need to learn to perform everyday tasks. The mathematician sees mathematics as a thinking discipline that is ever-changing with precise universal language and patterns.

Differing Opinions of Excellent Mathematics Teaching

There was a division among scholars, researcher, mathematicians, and mathematics educators as to what excellent mathematics teaching is when this study was designed. During the 1990s, the teaching of mathematics became the subject of headed controversies known as the math wars (Schoenfeld, 2004). "While the "math wars" in the U.S. (and now in parts of the world as far distant as Israel) are in a sense outside the realm of educational psychology (though not social psychology) and research in mathematics education, they are a critically important phenomenon that needs to be discussed. Researchers need to understand the context within which their work is done (Schoenfeld and Pearson, 2009).

Two strongly opposed views of what is considered excellent elementary mathematics teaching can generally be considered having conceptual and procedural epistemologies. The conceptual view, in general, is based on constructivism and Piaget's theory of mental stages of development. This theory involves assimilation and accommodation of the cognitive structure in the brain of the individual. The biological growth as well as the experience that a person has had influences his ability to assimilate or accommodate a learning experience. This theory has been most prevalent in the last few decades. The procedural view, in general, is based on behaviorism and Skinner's theory of stimulus-response. This theory believes that students learn from external variables and is controlled by the situation, the behavior, and the consequences of the behavior (Bell, 1978). This theory was most prevalent during the 1950s through the 1970's and gave rise to empirical studies in education (Cottrill, 2003). Behaviorism and Constructivism, as well as Piaget and Skinner, are not to be considered as exclusive theories and epistemological underpinnings of mathematics education. They are used here to generally define opposing views of what is considered to be excellent mathematics teaching and learning.

The constructivist believes that excellent mathematics teaching involves problem-solving, communication, reasoning, using manipulatives and calculators, making connections and invented algorithms. This type of teaching is supported by current standards-based or reform teaching (NCTM, 2000). This type of teaching is contrary to and challenges the ideas of "content-oriented" view of mathematics that predominated for more than a century (Scheonfeld, 2004). The procedural view of mathematics focuses on drill and practice and memorizing facts, formulas, and procedures that are based on theorems and proofs that are universal and can be used in more advanced mathematical studies. This view focuses less on social interactions and more on paper and pencil calculations. Many mathematicians subscribe to this view because of the nature of their discipline. Fluency, precision and accuracy of mathematical terms, procedures, and notations are important. It should be noted that mathematicians are also concerned with "relational understandings" (Skemp, 1978). This kind of understanding of mathematics includes knowing what to do and why.

The following discussion on constructivism and later on behaviorism is not to be considered a complete definition of each term. A discussion on opposing views of mathematics education is used here in an attempt to understand what was considered excellent mathematics teaching identifying the roles different philosophies had within these paradigms in literature by scholars and theorists during the early part of the twenty-first century. It is important to note that a more "balanced" approach is advocated today as a result of the collaboration and research that evolved from this controversy.

Constructivist view on excellent mathematics teaching. The constructivist believes in the cognitive learning theories of Piaget and Vygotsky. Learning takes place based on the available schema or cognitive structure of the child. This developmental stage theory suggests that the ability to learn something is determined by experience the child has had previously and the child's stage of development. The child tries to make sense of the learning experience by using assimilation by comparing the cognitive structure and the physical environment to prior experiences and accommodation modifying the cognitive structure (Hergenhahn & Olson, 1997). This cognitive process is internal and individual. Piaget (1973) suggested that individuals actively construct knowledge internally through their actions on objects in the world and their reflections on these actions. Vygotsky's theory contains elements of social interaction. According to Vygotsky (1978), individuals construct knowledge in the zone of proximal development. He describes it as the distance between the level of development of a student (working on a problem individually) and his or her level of potential development (working with an adult or teacher). This zone allows the adult to be the "tool holder," that is, having conscious control of the concept, for the child until he or she is able to internalize external knowledge (Cottrill, 2003). Vygotsky believed that individuals could achieve *higher ground* through talking; that is, through interactions individuals organize their thinking and actions. In this way, learning occurs two times for each individual, originally during social interaction and then again within the individual (Steele, 1995). The constructivist believes that the learner must have an active role in the construction and understanding of knowledge to learn. The role of the teacher in constructivist theory is to "facilitate" or "guide" the student to discover knowledge.

Behaviorist view on excellent mathematics teaching. Behaviorism is guided by the theories of Skinner. This theory attempts to explain learning through the observable interactions of the learner with the environment (Cottrill, 2003). This funtionalistic theory involves relationships between stimuli and response. Skinner believed that stimuli can be manipulated to achieve a specific response. Therefore, learning can be accomplished by forms of reinforcement and punishment. Skinner (1958) described this programmed learning for education in an analogy of a machine and tutor as follows:

> The machine itself, of course, does not teach. It simply brings the student into contact with the person who composed the material it presents. It is a labor-saving device because it can bring one programmer into contact with an indefinite number of students. They may suggest mass production, but the effect upon each student is surprisingly like that of a private tutor. The comparison holds in several respects. (i) There is a constant inter-change between program and student. Unlike lectures, textbooks, and the usual audio-visual aids, the machine induces sustained activity. The student is always alert and busy. (ii) Like a good tutor, the machine insists that a given point be thoroughly understood, either frame-by-frame or set-by-set, before the student moves on. Lectures, textbooks, and their mechanized equivalents, on the other hand, proceed without making sure that the student understands and easily leave him behind. (iii) Like a good tutor, the machine presents just that material for which the student is ready. It asks him to take only that step which he is at the moment best equipped and most likely to take. (iv) Like a skillful tutor, the machine helps the student come up with the right answer. It does this in part through the orderly construction of the program and in part with techniques of hinting, prompting, suggesting, and so on, derived from an analysis of verbal behavior . . . (v) Lastly, of course, the machine, like the private tutor, reinforces the student for every correct response, using this immediate feedback not only to shape his behavior most efficiently but to maintain it in strength in a manner which the laymen would describe as "holding the student's interest." (p. 971)

Behaviorists fundamentally believe differently about how a person learns than constructivists. Learning occurs, for a behaviorist, by external manipulation without any reference to anything that is going on inside the learner. Skinner's theory has been called "the empty organism approach" (Hergenhahn & Olson, 1997). In the education community it has been called the empty vessel theory. It is thought that the teacher, in this theory, must fill the vessel (child's brain) with the pertinent knowledge. Direct instructional techniques are based on this type of learning theory. The role of the teacher is to present information and have students respond giving immediate feedback on the correctness of the response. Students are to attend to the information in a way to be able to respond back to the teacher to get a correct answer. It should not be assumed here that behaviorists are not concerned with thinking strategies or understandings or that constructivist are not concerned with giving students information or answers.

Seeking Common Ground

As the need for proficient mathematical skills for our citizens has escalated in the 21st century world, the solution(s) about what constitutes excellent elementary mathematics teaching continues to be confusing and controversial. Although there are strong opposing views, behaviorism and constructivism, about what excellent elementary mathematics teaching should be there is a current movement to find common ground among scholars, researchers, mathematicians, and mathematics educators (Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmidt, & Schaar, 2005; Schoenfeld, 2004). Scholars realize that without a common ground of understanding of what is considered excellent elementary mathematics teaching efforts to improve mathematics education in public school classrooms will not reach its potential. Some scholars contend that it may even make the mathematics education of our children worse (Schoenfeld, 2004; Skemp, 1978). Some scholars suggest that disagreements between different views might be more a matter of language and the lack of communication than representative of fundamental differences of view (Ball et al., 2005). The academic community has reached a common ground that could possibly guide an effective movement that could improve mathematics education in the United States.

My Position on the Controversy

I do not take an either or assumption to supporting one side of the controversy or the other. As a teacher, I was concerned about both understanding of mathematical knowledge and my student's ability to compute properly. I believe most, if not all, educators want both procedural and conceptual mathematical knowledge for their students. A teacher's philosophical beliefs may influence their classroom practice more than a specific reform movement. A teacher with either a constructivist or a behaviorist philosophical underpinning can teach elementary mathematics well. One who uses both epistemological teachings has the potential to teach mathematics more effectively to more students. The more important question is: What does excellent elementary mathematics teaching look like in the classroom and how can we combine this information with what we already know and have learned throughout history to effectively lead our elementary schools to more success in the teaching and learning of mathematics? The study of quality practice could lead to deeper understanding of theory.

One of the assumptions in this study is that teachers are a catalyst of change. Teaching of mathematics in the United States can only improve with the contributions and efforts of teachers who are the leaders of practice in today's classrooms. Studying excellent teacher practice can provide a missing link that can bridge theory to practice and show what excellent mathematics teaching looks like in an elementary classroom. This could lead mathematicians, mathematics educators, teachers, parents, politicians, and college teacher preparation programs to a better understanding of how to improve student achievement and the teaching of mathematics in the United States and around the world.

Problem Statement

Today, it is critical that all of our children have mathematical proficiency to have a chance to be productive and successful citizens. According to the National Research Council (NRC, 2001), mathematical proficiency consists of the following strands:

1. *Conceptual Understanding* – comprehension of mathematical concepts, operations, and relations
2. *Procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
3. *Strategic Competence* – ability to formulate, represent, and solve mathematical problems
4. *Adaptive Reasoning* – capacity for logical thought, reflection, explanation, and justification
5. *Productive Disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

The National Research Council also stresses that these five strands are interwoven and interdependent. Results of state, national and international assessments for the last 30 years have shown that our children are not achieving mathematical proficiency. The performance of United States Children in mathematics is considered "unacceptable" (CBMS, 2001; NRC, 2001; Rand, 2003; Stigler & Hiebert, 1999). The academic and professional community offers multiple views of what excellent elementary mathematics education is and should be. These opposing positions complicate the goal of improving student achievement and teacher's teaching of elementary mathematics in the United States. We are sure that our children are not achieving the level of mathematical proficiency that is needed, but there seems to be no consensus as to what excellent elementary mathematics teaching and learning looks like in the classroom.

Purpose of the Study

The purpose of this collective case study was to describe what three purposefully-selected excellent elementary mathematics teachers who positively affect student achievement in public elementary schools in the Midwestern United States know and what they did in mathematics instruction. This study informs us more clearly about the competing ideas of what excellent mathematics teaching is and what excellent elementary mathematics teachers decided to teach and how they taught it. Data were collected during three semi-structured interviews and nine plus consecutive classroom observations of each teacher and included a teacher Content Knowledge for Teaching Mathematics Measures (CKT-M) survey during one semester of one school year. This study did not directly study the multicultural or socio-economical issues of teaching mathematics. It also did not directly study specific curriculum materials or what mathematical content the curriculum should have. It briefly outlined how these teachers found and used curricular materials and their beliefs about textbook usage in the teaching and learning of mathematics. Outside influences of the practices of these teachers were not studied but the study examined how these excellent teachers interacted with these stakeholders. The focus of this study was on the excellent teachers and their teaching, what they did, and what they knew in elementary mathematics.

Study Questions

Grand tour question. How do three excellent, public school, elementary mathematics teachers in a Midwestern state teach mathematics in their classrooms? What did they know and what did they do?

Research questions.

1. How do three excellent elementary mathematics teachers choose to deliver instruction?
2. How do three excellent elementary mathematics teachers perform on a Mathematical Knowledge for Teaching (CKT-M) instrument?
3. How do three excellent elementary mathematics teachers choose representations, examples, terms, and explanations for students? Why?
4. How do three excellent elementary mathematics teachers analyze student understandings of mathematical ideas and concepts?
5. How do three excellent elementary mathematics teachers monitor and assess students' mathematical understandings?
6. How do three excellent elementary mathematics teachers develop their skills and knowledge to teach mathematics?
7. What do three excellent elementary mathematics teachers know and believe about the learning of mathematics?

These introductory questions evolved into key issues (Stake, 1995) and a more important question arose during the study from what excellent teachers do and know to an exploration of what excellent teaching is based on this case. Stake suggested that, "The best research questions evolve during the study" (p. 33).

Limitations of the Study

This study focused on purposefully-selected elementary teachers in one mid-western state, during one semester of one school year. Only three excellent elementary mathematics teachers were studied in detail. The study consisted of 9 to 12 consecutive observations of one unit of math teaching of each teacher. The confirmability of the study was done through member checks and an audit conducted by a third party.

Significance of the Study

Many studies of mathematics teaching have been conducted. These studies were generally conducted for specific understandings and comparisons mostly with a functionality component. For instance, how does a treatment (professional development program) influence practice? Or how does one program or set of teachers compare with another? Stigler and Hiebert (1999) studied video taped teachers in three countries attempting to distinguish differences in the mathematical teaching methods of the different cultures. Ma (1999) found that United States teachers taught and understood mathematics differently from teachers in Japan. Schifter and Fosnot (1993) studied the impact of a summer teacher development program on meeting the challenges of reform. Steele (1995) studied a fourth grade teacher in mathematics teaching using qualitative methodology focusing on a constructivist approach to teaching.

Literature is lacking in studies of excellent elementary teacher's teaching of mathematics and how that relates to the theoretical underpinnings. This study did provide a detailed description of excellent mathematics teaching from a new lens. It described what is right and effective in mathematics education that could contribute to studies about what is puzzling with mathematics instruction in the United States. It described excellent elementary mathematics teaching without allegiance to a specific theoretical hierarchical position.

By examining these excellent elementary mathematics teachers' teaching using qualitative approaches, we can better understand what excellent elementary mathematics teaching looks like in the classroom. With this understanding, mathematics educators and mathematicians can better formulate and develop models of teaching elementary mathematics that involve practice as well as theory. Researchers can better isolate variables and develop new models for studying what excellent mathematics teaching is. Administrators and teachers can plan interventions and show models of excellent teaching of mathematics in real classrooms that may change attitudes towards one's own ability to teach math. Parents can better understand mathematical practices and assist their children in learning mathematics.

Chapter 2

Methodology

Rationale for a Qualitative Study

The United States, as well as other countries around the world, has struggled to improve the performance of their children in mathematics achievement in schools. Many attempts through reform movements have been made over the years to address these shortcomings. The results of these efforts have been “baffling” (Cohen, 1989; McLaughlin, 1990; Sarason, 1971; Skemp, 1989; Tyack & Cuban, 1995), and “unacceptable” (Rand, 2003; NCMST, 2000; NRC, 2001). Differing views of what excellent elementary mathematics teaching is have changed reform perspectives throughout history. One decade will focus on procedural mathematics, another on curriculum, and another on conceptual understandings of mathematics. Recent reform efforts have included the specialized knowledge needed in the teaching of mathematics led by Deborah Ball of the University of Michigan (1988, 1990, 1991) and others (e.g., Ball & Bass, 2003; Ball & Cohen, 1999; CBMS, 2001; Cohen, 1989; Lampert, 2001; Shulman, 1987).

The focus of the study was to understand and describe what excellent elementary mathematics teaching and learning looked like in the classroom by selected excellent elementary teachers. Qualitative research methodology was currently the best theoretical approach to this study. Creswell (1998) defines qualitative research as follows:

> Qualitative research is an inquiry process of understanding based on distinct methodological traditions of inquiry that explore a social or human problem. The researcher builds a complex, holistic picture, analyzes words, reports detailed views of informants, and conducts the study in a natural setting.

This study explored a human issue in its multiple complexities and dimensions. The complexities of excellent mathematics teaching and learning included: teaching, learning, mathematics, knowledge of students, and specialized knowledge needed to teach the discipline. The dimensions of this case were mathematical, historical, philosophical, theoretical, pedagogical, social, and personal.

One century ago, philosopher Wilhelm Dilthey argued that science was not moving in the direction of helping humans understand themselves:

> Only from his actions, his fixed utterances, his effects upon others, can man learn about himself; thus he learns to know himself only by the round-about way of understanding. What we once were, how we developed and became what we are, we learn from the way in which we acted, the plans which we once adopted, the way in which we made ourselves felt in our vocation. . . . We understand ourselves and others only when we transfer our own lived experience into every kind of expression of our own and other peoples lives. (Dilthey, 1976, p. 163)

I sought to understand the complex interrelationships, holistically, among all that existed through description and interpretation. I personally captured the experience so, I could interpret it, recognize its contexts, puzzle the many meanings while still there, and pass along an experiential, naturalistic account for readers to participate themselves in some similar reflection (Stake, 1995). The qualitative paradigm allowed me to study the humanistic work of teaching in all of its complexities.

This study needed to be conducted in elementary classrooms to gain a naturalistic experience of what excellent elementary mathematics teaching is and looked like. The teacher's voice is critical for the in-depth understanding of what excellent teaching is. By "telling the story" of excellent teachers actually teaching mathematics we can come to understand the unique expertise that constitutes excellence in the complex execution of the profession of teaching, learning, and the discipline of mathematics.

I believed that our current understanding of excellent elementary mathematics teaching is incomplete in some way. Scholars, researchers, mathematicians, and policymakers, among others, believe that change in classroom practice is imperative to improve the teaching and learning of mathematics in our schools. Qualitative research provided a framework to face changes in our educational practice, to develop new intellectual perceptions and to change the structure and condition of our educational setting (Denzin & Lincoln, 1994). This study could influence the understanding of what excellent mathematics teaching and learning is and how to implement new structures for future generations of teachers and students based on the experiences and practices of the excellent teachers in this study.

Qualitative Assumptions

In this study, and most qualitative studies, reality is subjective and multiple. There are multiple perspectives from participants, researchers, readers, etc. Reality is constructed by individuals involved in the research situation (Creswell, 1998). I used quotes in the words of the participants and provided evidence of multiple and/or different perspectives. "These ontological assumptions support experientially rather than operationally defined variables and allowed for flexibility in the direction the study took based on the data collected. Subjectivity is not seen as a failing needing to be eliminated but as an essential element of understanding" (Stake, 1995, p. 45).

In qualitative research, the researcher is the instrument (Stake, 1995). I interacted with the people and places in my study. I conducted observations and interviews and focused primarily on the participants in this study. I tried to minimize the distance (Lincoln & Guba, 1985) and become an insider (Creswell, 1998) between myself and those being researched and my goal was to be noninterventionistic (Stake, 1995). I became an insider by the consistent unobtrusiveness in my extensive observations in each classroom. My values were explained and monitored in this study considering my participatory position. I admit that the research is value laden and that biases are present and will openly discuss them in the narrative. “We (re)present our data, partly based on participants’ perspectives and partly based on our own interpretations, never clearly escaping our own personal stamp on the study” (Creswell, 1998, p. 20).

I used a personal and literary narrative and specific terms of the qualitative paradigm in this study. I used terms such as credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985) instead of terms such as internal validity, external validity, generalizability, and objectivity. The language of my study is personal and based on definitions that evolved during the study rather than those defined at the beginning. My writing gives the reader a vicarious experience of “being there” and that the data would be the same had I not been at the site.

This study used inductive logic, studying the cases within their context, and used an emerging design based on previous data. I worked from particulars to generalizations, described the context of each participant, and continuously revised interview questions and focuses of observations from experiences in the field. Dimensions, sometimes called themes, were developed by “layering the analysis,” (Creswell, 1998) by presenting key issues for each participant initially, followed by cross-case grouping of these key issues into broader and more abstract categories and later leading to a set of dimensions. This process is described in detail later in this chapter. This inductive process of developing dimensions from the participants in the field rather than specifying them in advanced was utilized during this study.

Type of Qualitative Design- Case Study

I chose a collective case study methodology to conduct my research. I purposefully selected excellent elementary mathematics teachers to develop a deep understanding of what excellent mathematics teaching is and looks like in an elementary classroom. A case study is an exploration of a "bounded system" or a case (or multiple cases) over time through detailed, in-depth data collection involving multiple sources of information rich in context (Creswell, 1998). This study was "bounded" by time and place. Data was collected in April and May of the 2007-2008 school year in one mid-western state. Each participant was studied consecutively and took up to three straight weeks for each teacher. It occurred only in elementary classrooms and only studied the teaching of mathematics. It was a "collective case study" (Stake, 1995) because I studied more than one teacher (case). This study is a multi-site study. It studied each participant in her own classroom. In case study the context of each case is important. "The context of the case involves situating the case within its setting, which may be a physical setting or the social, historical, and/or economic setting for the case" (Creswell, 1998, p. 61).

This case study was an instrumental case study (Stake, 1995) because I used the cases instrumentally to illustrate an issue. I conducted this study to describe and understand what excellent elementary mathematics is and looks like in a classroom. Multiple sources of data were collected to provide a rich, thick description with multiple dimensions and realities providing a holistic view. Data were collected through interviews, observations, researcher's notes, and a CKT-M survey. The study generally investigated how teachers taught mathematics (observations) and how and why they made decisions about mathematics teaching (interviews). The study included within-case analyses to provide an embedded exploration of each specific case and a cross-case analysis to create a holistic view of the case.

The intent was to make it possible to understand the complex system of teaching elementary mathematics by the illumination and investigation of teaching events conducted by successful teachers. This case study methodology allowed for an emerging design (Creswell, 1998; Stake, 1995) where questions and research focus changed during the process of research based on the data to reflect an increased understanding of the case. The direction of the study focused on how to best learn from the participants by describing and interpreting the meaning of their classroom experiences. Stake (1995) describes the best rationale for using a case study methodology:

> The real business of case study is particularization, not generalization. We take a particular case and come to know it well, not primarily as to how it is different from others but what it is, what it does. There is emphasis on uniqueness, and that implies knowledge of others that the case is different from, but the first emphasis is on understanding the case itself (p.8).

What is needed in literature and the profession is a developed understanding of what excellent elementary mathematics teaching is and how it operates in practice. These participants may be unique and different from other teachers but may be able to provide a deeper understanding of the successful practice of teaching elementary mathematics.

Role of the Researcher

I bring educational experiences to my role as a researcher in this study. My background includes five years of teaching experience at the elementary level. I have taught elementary mathematics methodology classes to pre-service teachers at the collegiate level. I have also supervised pre-service teachers in mathematical practice and student teaching. I have also completed extensive learning experiences in pedagogical skill and knowledge of the work of teaching elementary mathematics.

In this study the researcher was the main instrument of data collecting and analysis. I was immersed in the natural setting, conducted the interviews, and observed the observations with my own eyes. In this way, I become an "insider" in the system. My intent was to be a "noninterventionist" (Stake, 1995) seeking naturalistic observation. My goal was to see excellent elementary mathematics in action as if I were not there. I did not participate in the day to day operation of the teaching of mathematics. My role was to describe and experience the real life operation of excellent mathematics teaching to understand what it is and what it looks like. I used a protocol and audio taped each observation, conducted individual audio-taped interviews with each participant, and administered a CKT-M survey to each teacher in her classroom.

Stake (1995) also talks about the researcher's role within case studies. He suggests researchers as: teacher, advocate, evaluator, biographer, interpreter, and roles in constructivism and relativity. He also insinuates that these roles change for specific situations and that the researcher can not be separate from the research. "The researcher is the agent of new interpretations, new knowledge, but also new illusion" (p. 99). I realize that this research is value-laden and is not bias-free (Creswell & Miller, 1994). I acknowledged the disciplined subjectivity of Erickson's (1973) rigorous self monitoring, continuous self-questioning and reevaluation of all phases of the research process. I openly expressed my biases in writings, journals and notes. As I played some of the roles that Stake (1995) mentions, throughout the study, I was supportive of the needs of the participants and became as neutral as possible. I anticipated the roles of interpreter, evaluator, and one that is not included, observer. I refrained from being an advocate or teacher as I sought to understand the participants and their practices, beliefs, and experiences.

Participant Selection

The quality of the participants was critical in this study and purposeful sampling was used to best understand the case. Schumakcher and McMillan (1993) describe purposeful sampling as “selecting information-rich cases for study in-depth.” An extensive search to find the best teachers was conducted. The goal was to study three excellent elementary mathematics teachers. I conducted a search for possible participants (teachers) by asking for nominations from multiple reputable sources of mathematics and mathematics education leaders (see appendix A for nomination letter and form). The following is a list of sources that were contacted to receive nominations for possible participants:

Nebraska Department of Education
Nebraska Council of School Administrators
National Council of Teachers of Mathematics
Nebraska Association of Teachers of Mathematics
School Administrators
School District Math Specialists
College and University Teacher education programs
ESU (Educational Service Units) Staff Developers

I asked these sources to provide a list of teachers that were considered the best elementary mathematics teachers. I provided the sources with a list of prerequisite criteria in the selection process. The following is a list of the prerequisite criteria that needed to be met before nominating a candidate:

1. The candidate must be a practicing teacher in an elementary school (K-6).
2. The candidate must teach and be selected for his or her mathematics teaching.
3. There is evidence that the teacher positively affects student achievement in mathematics.

Three teachers were selected from rural, urban, and suburban schools to show the impact of specific settings on mathematics teaching is and looked like and what patterns were evident among excellent teachers. This strategy provided depth to the case and access to mathematics teaching methods for students of a variety of ages and geographic locations. My goal was to compare names of excellent teachers from multiple sources. I took the 21 nominations and developed a list of possible participants starting from those I determined to be the most potential based on the criteria provided to the nominators. I made a list of the 21 nominations received in order from the top candidate down and contacted school administration to receive permission to contact candidates. I then contacted candidates from the top spot down until 3 participants voluntarily accepted to participate in the study. The top three candidates all accepted the invitation to participate in the study. These volunteer participants signed an informed consent form required by IRB and their schools were located within 100 miles of the researcher's location (see Appendix B).

Ethical Considerations

The participants' rights, interests, and sensitivities of this study were considered first. Participation was voluntary and the participants were allowed to discontinue participation in the study at any time without reprimand. Participants remained anonymous by the use of pseudonyms and access to data was confidential only to the researcher and each individual participant. Data collected from one participant were not available to the other participants. I thoroughly explained my research design to each participant and sought out and prioritized participants' views, beliefs, and ideas throughout the study.

Reciprocity was used with participants. This meant that each participant collaborated with and responded to the researcher's collection of data and its interpretations through "member checks" (Creswell & Miller, 1994) and had opportunities to provide feedback and disconfirming evidence of my description and interpretation at all phases of the study. I did not receive any disconfirming feedback from any of the member checks. Each participant received a final copy of the research study.

Participation in the study could help the profession understand better what excellent elementary mathematics teaching is and looks like and could help participants learn more thoroughly about their own practice. I obtained IRB approval from the necessary review board before contact occurred with participants and protected participants from physical or mental discomfort or harm (see Appendix C). The participants were selected for their excellence and value to the profession and their potential to provide knowledge of the teaching and learning of mathematics.

Data Collection

Data were collected during the spring semester of the 2007-2008 school year and took approximately ten weeks to complete. The multiple sources of data that were collected included interviews, observations, researcher's notes, and a teacher survey (CKT-M). The data collection process was constructed to maximize the potential of understanding the complexities of the case. I conducted interviews after observations asking questions to clarify interpretations and understand the participant's intent and thinking about situations or events. The survey was administered before the last round of interviews and observations to provide adequate time for discussing the results and to minimize potential bias of each teacher's authentic teaching.

Each participant's schedule and commitments were accommodated. The data were collected for each participant solely and occurred in succession throughout the spring semester of one school year. Data were briefly analyzed throughout the collection process to guide my focus and attention of future interviews and observations. This was not the final analysis of the data but analysis to direct the next phase of data collection for each participant. All protocol decisions, questions, and copies were collected for the audit trail and for triangulation verification and approved by IRB.

Observations

I conducted a minimum of nine consecutive observations of each participant in her classroom during the teaching of mathematics. I used an observation protocol (see Appendix D) that was adapted from models of Stake (1995) and Schumacker and McMillan (1993) and recorded information that was both descriptive and reflective (Creswell & Miller, 1994). Generally, field notes taken during the observations were descriptive and entries after observations were reflective. I consulted the participant before the observation to get a brief understanding of what would be occurring during the observation. I conducted only one observation on a given day and set aside time immediately after each observation to reflect on the experience and record reflective field notes.

My goal of the observations was to see what mathematics teaching looked like in each participant's classroom. I used the data from observations to describe what each participant did in her teaching including verbatim classroom interactions of mathematics teaching and learning. This component of data collection helped me understand and see transitions from theory to practice of elementary mathematics.

The first observation was broad and included the contextual description of the setting, participants, procedures and physical diagrams. The subsequent observations sought data that focused on answering the study questions and to "reveal the unique complexities of the case" (Stake, 1995). The naturalistic setting provided experiential data that can only be obtained from real life experience. I sought to conduct observations that Stake (1995) states "keeps a good record of events to provide a relatively incontestable description for further analysis and ultimate reporting. He or she (the researcher) lets the occasion tell its story, the situation, the problem, resolution, or irresolution of the problem (or issue)" (p. 62).

Interviews

In qualitative studies, researchers use open-ended or semi-structured interviews to collect data (Creswell & Miller, 1994). This study used open-ended questions in a semi-structured way. That means the questions were formulated in an open-ended way that "invites the participant to participate in an in-depth way." These conversational questions helped me to learn as much as I could from the participants and allowed their own language and perceptions to emerge. These open-ended questions used a semi-structured format. Some of the questions were pre-determined but other questions evolved and changed throughout an interview and between participants throughout the study to best understand the case. Probes were used during interviews with participants (Creswell & Miller, 1994). Both clarification and elaboration probes were used to understand the participants. Each participant received a list of possible questions before the interview. A protocol (see Appendix D) was used for each interview to record participant responses. All the interviews were audio-taped and transcribed verbatim for data analysis using Creswell and Miller (1994) guidelines.

The interviews were used to collect data from each participant that "will provide information unavailable through observation" (Bogdan & Biklen, 1992) and to seek deeper understanding of the thoughts and perceptions of each participant. Each interview was face-to-face with each participant in her own classroom. The first interview was more demographic in nature. I was able to get to know the participant's philosophy, history, background, etc. and I was able to get to know the participant personally. The subsequent interviews focused on data for the study questions and understanding of previous experiences during observations. These protocols were developed as the study evolved. The second protocol was prepared after a few more observations and before the second interview. The third protocol was prepared after a few more observations and the survey. Protocol's questions emerged during the study and slightly varied by participant to develop an in-depth rich understanding of each participant's beliefs and experiences. "Qualitative case study seldom proceeds as a survey with the same questions asked of each respondent; rather, each interviewee is expected to have had unique experiences, special stories to tell" (Stake, 1995, p. 65).

CKT-M Survey Instrument

One of the concerns about elementary mathematics teaching in the profession and in literature is the content knowledge or subject matter knowledge of the teacher. Evidence suggests that United States teachers lack critical knowledge for teaching topics in mathematics (Ball, 1990; Ma, 1999; Stigler & Heibert, 1999). Language in governmental documents, such as the No Child Left Behind Act, and commission documents, such as "Before It's Too Late," "Nation at Risk," "Mathematical Proficiency for All Students," "The Mathematical Education of Teachers," and others show concern for "qualified teachers." In the past decade, attempts to understand what a qualified teacher is and how to measure it have developed. The Study of Instructional Improvement (SII) at the University of Michigan in 1999 began to design measures of K-6 teachers' knowledge for teaching mathematics. This survey instrument has been referred to as CKT-M survey throughout this study (see published examples of instrument in Appendix E). It stands for Content Knowledge for Teaching Mathematics Measures. I attended a mandatory workshop at the University of Michigan on the use and development of the instrument to obtain approval and expertise to use it in this study.

The instrument was designed by current teacher knowledge theory based on Shulman (1986, 1987) and colleagues (e.g., Wilson, Shulman, & Richert, 1987). Shulman proposed three categories of teacher subject knowledge (Hill et al., 2004). The first category, *content knowledge*, for Shulman was defined as including both facts and concepts in a domain, but also why facts and concepts are true, and how knowledge is generated and structured in the discipline (Bruner, 1960; Schwab, 1961/1974). The second category is *pedagogical content knowledge*. Shulman went "beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching." Shulman (1986) includes representations of specific content ideas, as well as an understanding of what makes the learning of a specific topic difficult or easy for students. The third category, *curriculum knowledge*, involves using curriculum materials, and knowledge of what a teacher will teach in his or her class but also what is taught and needed throughout a student's career studies in that domain. This suggested that the knowledge needed to teach a subject well included content knowledge and specialized knowledge needed to teach such content.

Using Shulman's theory, four goals guided the designing of items for the survey (CKT-M, 2005):

1. To reflect the knowledge teachers use in teaching—both the content they actually teach students, and the special knowledge teachers have in order to teach this subject matter to students;
2. To situate these problems in the contexts the teachers face in classrooms-examining textbook definitions for accuracy, designing classroom tasks, evaluation student statements;
3. To write items that did not represent any particular view of how mathematics should be taught;
4. To write items that would discriminate among teachers.

Items were developed in common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content, curriculum, and teaching. These questions became a catalyst for discussions with the teachers about their beliefs, philosophies, and teaching and learning decision making processes. The form of the instrument that I used for this study, 2001A version, uses the domain of number and operation.

The Content Knowledge for Teaching Mathematics (CKT-M) instrument was used in this study as a sampling procedure to provide confirmation that the selected teachers had excellence in elementary mathematics teaching. I did not use this quantitative data in the analysis of the case beyond the qualification that the results indicated that the selected teachers showed signs of excellence. Discussion with each participant regarding their performance on the instrument is presented in chapter 3. Quantitative data were not used in the analysis procedures in chapters 4 and 5. The survey instrument became a catalyst for discussions and reflections from the teachers about teaching mathematics and personal experiences with learning mathematics that possibly would not have been discovered without its use and was not anticipated in the design phase of the study.

Data Analysis Process

Data analysis included a detailed description of the case and its setting (Creswell, 1998) to present a naturalistic and vicarious experience. Data analysis and data collection were simultaneous activities rather than discrete activities (Creswell & Miller, 1994; Stake, 1995) during this study. I developed early codes from the data that were tentative and guided focus on further data collection. The analysis of the data in this study was “recursive, dynamic, and flexible” (Merriam, 1988) and included both within-case and cross-case analysis. I collected extensive data from multiple sources in multiple forms and conducted a holistic and embedded analysis of the data. A critical task that I faced in the analysis process was to identify the best data that portrayed the case and set aside the rest. Harry Wolcott (1990) wrote that the critical task in qualitative research is not to accumulate all the data you can, but to “can” (i.e., get rid of) most of the data you accumulate. I used “winnowing” techniques to seek the best data keeping the case and the study's questions in focus. What Stake (1995) calls correlation or covariation in quantitative study, he calls pattern and correspondence in qualitative study.

I analyzed the observations, interviews, and field notes of each participant individually at first. The observations were audio taped and highlighted events transcribed. The interviews were audio taped and transcribed verbatim. The audio tapes were also used throughout the analysis process to re-experience events and present verbatim interactions between the teacher and students that contributed to the meaning of the case. The field notes were used in analysis and provided additional meaning but were not coded. The "in vivo" coding process of Creswell and Miller (1994) was used to analyze these data. First, text codes were identified. Text codes were usually a single term or word or phrase that represented the meaning of a small amount of text. Next, the text codes were clustered by similar meaning and labeled by what that cluster was about holistically. The clustered codes were used in part to select teaching episodes and used in the cross-case analysis. I started with approximately 250 text codes that were reduced to approximately 20 cluster codes through a process of "dross" and reduction of text codes that did not fit (Miles & Huberman, 1984) or that were overlapped and redundant (Creswell, 1998). Teaching episodes were selected for each participant for in depth analysis and represented the clustered codes and everyday teaching occurrences. Key issues for each participant's teaching were identified by searching for correspondence, patterns, and meaning of the episodes. This was a process that Stake (1995) calls "progressive focusing."

Additional Data Source: Related Literature

I then conducted a cross-case analysis to search for patterns and meanings from the participants holistically. This process started with comparing the key issues of each of the participants. I kept in mind data from the clustered codes and experiences I had in the classrooms looking for exemplar patterns of the three teachers (participants). At this point, I used the research literature as an additional data source making comparisons between theory and practice. I believed that using the research literature in this way was more effective and valuable than a separate chapter on literature review. This process led to a set of categories (exemplars) for the participants.

I chose to write an additional section on my interpretation of the unique blending of pedagogical and content skill of just one of the participants because of its significance to the understanding of the case. Stake (1995) states that "analysis and interpretation are the making sense of all this. How is this part related to that part?" I used "direct interpretation" (Stake, 1995) in addition to interpreting codes, clusters, key issues, cross-case categories, specific instances, and research literature to make sense of the data holistically. I looked for patterns to correspond data trying to understand behaviors, issues, and contexts with regard to my case.

I progressively focused on all the processes used in the analyses of the data to form eight dimensions of excellent elementary mathematics teaching from this case. The final phase of the analysis was to report, as Lincoln and Guba (1985) mention, the "lessons learned" from the case. Conclusions were developed based on the dimensions, what I learned from the participants about what excellent teaching is, and my reconsideration of the study questions. Implications to research and teacher education completed the "lessons learned" from this research case study.

Verification Procedures

It is important to discuss my paradigm position about qualitative study before I discuss my choices of verification procedures and how I used them. John Creswell and Dana Miller (2000) determined that there are two perspectives that govern the choice of validation procedures. These include the lens that researchers use to validate their studies and researchers' paradigm assumption. My paradigm assumption, in this case, is considered post-positivist or systematic. I believe that qualitative research consists of rigorous methods and systematic forms of inquiry (Creswell & Miller, 2000) actively employing procedures using specific protocols. I used multiple verification techniques for this study that included: member checks, triangulation, and an external audit. These verification procedures match Creswell and Miller's (2000) validity matrix. The member checks included the lens of study participants, triangulation included the lens of the researcher, and the external audit included an external lens of the study in the verification or validity process. Other validity processes were used in this study. I searched for disconfirming evidence, thick, rich description, researcher reflexivity, and collaboration with participants beyond member checks. This was all retained in an audit trail for the external audit.

Member checks. Member checks were conducted for each interview to seek participant's confirmation and perspective on data and its interpretation. These member checks were conducted informally through emails and notes taken in the researcher's journal. No major changes were provided by the participating teachers of the data that was presented to them during member checks through electronic mail.

Triangulation. In qualitative study where research is subjective, data analysis is deductive, and the researcher is the research instrument, I wanted to make sure that I got the description and interpretation of the case right. Getting it right meant that I chose accurate and credible descriptions and interpretations to the pertinent meanings of the case. I used triangulation, a systematic process of sorting through the data to find common themes or categories by eliminating overlapping areas (Creswell & Miller, 2000) as evidence of pattern development and understanding. This study allowed greatly for this verification strategy. I was able to triangulate what participants said (interviews) to what they did (observations) to understand the case. Disconfirming evidence and researcher reflexivity contributed to "getting it right."

External audit. This validity procedure brings an external lens to the verification process. It looked at the trustworthiness of the study from an outside expert to determine the validity of the process and product of the findings. The external auditor has achieved a doctoral degree that includes qualitative research. She is a former classroom teacher and currently is a professor of teacher education at the collegiate level. The audit examined the audit trail of the study and compared that data to the findings of the report. The external audit brought credibility to the logic I used in the development of findings, dimensions, interpretations, and conclusions of the data collected in this research study. An attestation of the audit and confidentiality agreement is included in Appendix F.

Chapter 3

Teacher Portrayals

Introduction

This chapter contains what I found from each participant and her teaching of mathematics through my experiences in each classroom. Three purposely selected excellent elementary mathematics teachers were selected for this study through nominations from educational mathematics leaders:

Sue Johnson, departmental mathematics specialist, 4th – 6th grades in a rural community school district
Kathy Freeman, 1st grade teacher in an urban city school district
Callie Hendricks, 3rd grade teacher in a suburban school district

The names of the teachers are pseudonyms to protect their identity but their school settings and teachings are factual. The results of each individual participant includes: the context, the teacher, survey results, observation and interview highlights, and concludes with key issues. I selected teaching episodes that represented clustered codes and everyday occurrences of each participant from her data. The key issues were developed by "correspondence" or finding meaningful patterns within all the episodes and observations of each teacher's teaching. These key issues turned out to be pedagogical and philosophical underpinnings of each teacher's beliefs about how students should learn mathematics (see Table 1).

Participant: Sue Johnson

Context. The drive to Sue Johnson's school was scenic and adventuresome. As I drove down the highway I saw fields ready for planting corn and beans, trees, birds, livestock, open spaces and occasionally a farm house. The one hundred mile trip was

Table 1

Participants Demographics

Name	Callie Hendricks	Sue Johnson	Kathy Freeman
Yrs. Experience	8	32	37
Yrs. Same School	8	32	36
Setting	Suburban	Rural	Urban
Students in Class	23	9	27
Physical Description	State of the Art	7 Yrs. Old	Over 30 Yrs. Old
Free/Reduce Lunch	13%	23%	26%

serene and a time for reflection. I saw only a few other drivers on the road, mostly tractors and trucks. It was not hard to find the turn into the small town where Sue taught. There was a farm house right at the turn into town with a few cattle grazing along the fence line.

Most roads in the town were paved but cracks and dips were evident. Road signs were scarce and outdated. Most of the homes in the community were two stories and well kept. The town of 500 had 3 blocks of downtown buildings with a gas station at the end of the linear strip. The gas station was the community hangout and had one older looking gas pump. Inside the gas station was a convenience store with an area for food options, snacks and tables where customers could sit and converse. As I walked into the gas station it was apparent that I was an outsider to the community. The three people in the store stared at me and seemed to be wondering who I was and what I was doing there. Many of the buildings in the downtown area were empty. Most of the remaining businesses were agriculturally related.

The community school system served the town and surrounding communities. There was a high school that was constructed in 1917 a couple blocks away from the elementary school that was constructed in 1999 and was the newest looking building in town. The all brick elementary school was tucked behind an older building that looked like an apartment complex. The front of the elementary school building had paved parking spaces and sidewalks with a gravel lot on the side of the building for additional parking. The playground was located in the back of the building and had dated school yard equipment that included an old metal swing set, a rusty merry go round, a spring horse, and three tether ball poles. The playground was surrounded by farmland that most recently was planted in corn.

The front entrance was the main way into the school and lead directly to the front desk and principal's office. To the left was a large, well equipped gymnasium where the students ate lunch and attended physical education classes. The hallways were carpeted with some lockers, windows, and student work covering the walls on both sides. Classrooms were equipped with modern amenities that included: televisions, whiteboards, cabinets, closets, tables, and other basic necessities. In the middle of the rectangular school structure was the library and newer restroom facilities that could be accessed in multiple directions.

Sue's classroom was roomy and full of bulletin boards, whiteboards, lists, cabinets, and tables. The walls were covered with color and helpful lists and procedures, mostly mathematical with some reading and language arts. A computer center that included a computer board was present in the front of the room. Several hand-held Palm Pilots were sitting on the back table. There were windows by the door to the hallway and the side of the room toward the playground. There were nine desks facing the front of the classroom. Each desk was connected to a seat with an opening under the top for books and supplies. There were books and paperwork lying on the floor by most of the desks. There were three rows of three desks with plenty of space between each desk. The teacher's desk was in the back of the room facing the backs of the student's desks. Sue had six boys and three girls in her 6th grade class and took a cart with mathematics materials to the 4th and 5th grade classrooms during specific times. Other teachers came to her 6th grade classroom to teach science and social studies. Sue also taught language arts, and reading as the 6th grade teacher. I only observed Sue teaching mathematics primarily to her 6th grade class. I observed one class period in the 4th grade and one in the 5th grade classroom during the study to see how Sue taught mathematics to students at the different developmental levels.

Teacher. Sue grew up in a small town and has lived her whole life in a rural community or on a farm. She met a family consumer science teacher in high school that influenced her to become a teacher of family consumer science. "That just seemed to be the logical route to go ahead in education" exclaimed Sue in an interview, "That is kind of what females did." "I enjoyed math and did well in math, but nobody really encouraged me to go into math."

Sue taught family consumer science at the high school for 22 years. The high school where she taught had students that were struggling in mathematics. Sue had a free period and volunteered to work with these students. She said she enjoyed teaching math and was able to help these students progress. These experiences rekindled her strong desire to teach mathematics. After 22 years of teaching high school family consumer science she jumped at the opportunity to become the departmental math specialist for 4th – 6th grades at the elementary school. She then enrolled in a master's degree program emphasizing middle school mathematics. "I wished I had taken more math classes in college," exclaimed Sue. It seemed to me that Sue has always had a love for teaching mathematics but didn't feel she had the support or background to teach it.

Sue Johnson has a bachelor's degree in family consumer science with a middle school endorsement in math, social studies, and family consumer sciences. She went on to earn a master's degree in teaching, learning, and teacher education with emphasis in mathematics. She is a member of various educational associations in education and mathematics. Sue Johnson is involved with professional development teaching in mathematics with a university and national professional development organizations. She designs programs and helps teach workshops on various mathematics concepts to in-service teachers around the state. Sue Johnson is a leader in her school district's mathematics curriculum and selected the textbooks that are used in all elementary classrooms. Sue has been a teacher for 32 years for the same district, 10 of those years as an elementary math specialist and 6th grade teacher.

Sue Johnson has attended professional development programs every summer to add to her teaching skill. She continues to look for more and better ways to study and present mathematics to her students. She told me that she looks at her educational service unit and colleges and universities for professional development opportunities where she can learn to develop multiple ways to present the mathematics to students in ways they can understand. Sue commits to at least one professional development program every school year. In our first interview Sue talked about the value of professional development,

> I can learn more or where I can add to something I have done or pick up something extra. Maybe it's a resource or a program idea or maybe it's a different technique. I'm not one to sit and be idle.

Survey. Sue Johnson took the CKT-M (content knowledge for teaching mathematics) instrument from the University of Michigan's Learning Mathematics for Teaching Project. Table 2 shows how Sue scored.

The IRT score is equivalent to standard deviations above the average score of the thousands of teachers that have completed the assessment. Note that in Patterns, Functions, and Algebra Sue scored all 12 questions correctly and has an IRT of 1.64. This means that of all the teachers that achieved a perfect score received the highest possible IRT score of 1.64.

Table 2

CKT-M Scores for Sue Johnson

Construct	IRT Score
Numbers & Operations	2.150
Patterns, Functions, & Algebra	1.640
Knowledge of Students & Content	0.305

Sue's scores in knowledge of students and content were surprising. From data that were collected during the observations I found her ability to know the relationships between students and content to be one of her strongest skill sets. When discussing this with Sue in an interview I found that the struggle was with the wording and the relationship of the question to her specific teaching situation rather than a lack of skill in the construct. For example, Sue missed 4 of the questions that asked for the "least likely," "most likely," "will likely have," and "best." These terms confused Sue because she teaches in ways that are spontaneous to the students' current understanding of an idea and not to a categorical valuing of those ideas. She also stated that, "When I teach it I talk about that and my students wouldn't make that mistake." Sue was thinking about how she teaches and related that to the question in an attempt to answer it in a categorical way.

The instrument's developers indicate that this construct (knowledge of students and content) scored the lowest in reliability and that other teachers had similar troubles with these questions ambiguity (Ball, Hill, & Schilling, 2004; IDW, 2005). Sue reported that she has had some previous experience working with questions and problems similar to those in this instrument. She felt that these problems were similar but a little easier than the ones that she worked on in her master's degree program. That is possible given that her training was in middle school mathematics teaching and this instrument was designed for elementary mathematics teaching.

Sue's teaching. In this section I integrated observations and interviews in my description of Sue Johnson's teaching. Also, within the writing of her teaching I will use examples during multiple lessons to show a more complete picture of her teaching. I observed nine consecutive math lessons of Sue's 6th grade classroom and one lesson on the 6th day of both the 4th and 5th grade math classrooms. I conducted three semi-structured interviews with Sue throughout the nine days of observations following the mathematics lessons during her planning time. I will discuss a series of teaching episodes that represent what she knows and what she does in her teaching of mathematics. They were chosen to depict a holistic view of her knowledge of the content, her pedagogical strategies, and her abilities to work with students in mathematics. The unit that she was teaching the sixth graders included lessons on cross multiplying, proportions, ratios, equations, perimeter and area, circumference, and work with trapezoids, triangles, and parallelograms. Mathematics in Sue's 6th grade class started in the early afternoon right after lunch and a brief recess. Mathematics was scheduled for one hour but frequently went longer. On occasion she devoted additional time for mathematics.

Each day I entered the classroom a few minutes early and was able to set up in the back of the room on a table while the students were at recess. I purposefully did not interact with the teacher or students throughout the observations because I wanted the experiences to be authentic as if I were not there. Students came in from recess and got a drink of water at a fountain in the classroom before they went to their desks to get ready for math. I noticed that none of the students were using a textbook. They were just quietly waiting for math to start. Sue teaches almost exclusively without a textbook. I asked her about it in an interview. She told me that the students have a "resource book" that is used to check for resources or answers, as a reference if a student forgot something or if a conversion was needed. I did not see the resource guide used to teach mathematics throughout my nine observations. Sue felt that she could teach mathematics better if she was not confined to a "scripted" lesson. She explained in the interview:

> I really steer from the text. Sometimes I don't even do things in the order that they do because I see some things to me that need to be done first. I don't think I always agree with the method or order that they're using. I have a better understanding of math. I feel that I don't have to use a scripted book because I know what I want to do. I find that sometimes I might feel like I get to where I want to go better when I work more spontaneously.

Sue also talked about tension that comes with not using a textbook. She told me that some parents feel frustrated at times because they struggle to help their child when there is no textbook and their child is being taught differently than they were.

> A lot of people (parents) were like what is this? This is not the way that I learned it. The other thing is that our math book is not a traditional math book where here is the lesson, here's the page of problems. For some parents that is probably difficult because they (students) take their math home and they don't take their reference book home and it creates problems because when parents are trying to explain to the kids there's no book there because I probably tend to use the reference book less than a lot of other teachers.

Sue told me that she informs parents at parent teacher conferences that students might be learning mathematics differently than they learned it. She also told me that other teachers in the school were uncomfortable with the inductive ways that she teaches mathematics and they didn't want to teach math in that way.

The first activity I observed was group work. This occurred in every lesson I observed and sometimes more than once. Sue wrote numbers on the board to distinguish who was in which group. The students got into three groups of three students. Each group gathered at one of the student's desk. The groups stayed the same throughout my observations except on one day when the students were reviewing for a quiz. Sue strategically changed one student per group for that day. She told me later it was because one group finished first all the time. I suspect that she also wanted to see how certain students could enhance other students understanding of the material for the quiz. The next day the students were back in the same groups that they had been previously. Sue talked about how she conducts group work.

> I constantly emphasize to them when we do group work that you need to work together and if they don't agree they have to talk about it. How did you do it? In that process they may discover where they made their error. I put them in groups not to grade papers. What were doing is checking our work to see if we understand

> it. Our goal is that everybody in that group needs to understand it better and its not meant to be a competition. It's about how is this going to help us all understand it better. Kids have to be more actively involved rather than be passive listeners. So by being in groups they have to be more accountable.

Sue did not use group work in the 4th grade or 5th grade classrooms in the one lesson I observed for each. I asked her about that in an interview.

> Fourth graders usually have the hardest time with group work partly because of their age and partly because they don't have a good expectation of what group work is and maybe do a little bit more arguing. Fifth graders are a little bit better and sixth graders know what is expected. There are always times when certain students should not be in the same group in any of the grades (levels).

Sue discussed with me reasons why she chose to use group work as a pedagogical strategy in her classroom and how group work helps her understand better what the students actually understand in an interview.

> I put them in groups and listen to what they're doing and saying and discussing. It's a way for me to have to hear what students are thinking because I walk around groups and hear them better. By putting them into groups and having them explain to each other using the vocabulary they were having to repeat what they have learned. By explaining it to somebody else then they started to have a better understanding. I have them verbalize more during class so I can hear what their vocabulary is. They have to verbalize it back to really understand it. What you think you say and what they understood are sometimes different.

Sue changed her teaching role dramatically where she could see and hear the students independently talking about the mathematics. This informed Sue about what she needed to do next to help her students' progress in the concept, skill or process.

In this next teaching episode note how Sue changes her teaching roles with group work. The task was to discuss proportion problems that had been assigned with each group. As the students talked within their groups about their problems, Sue walked around looking and listening to students talking about their work. She asked questions, made confirmations, and guided students' thinking based on what she was hearing and seeing. In their groups, students either agreed or disagreed with what other members of the group did or said and then would provide reasoning for their decision. After a while Sue brought the class back to whole class instruction and all the students went back to their desk. Sue demonstrated cross multiplying and proportions using an example of a problem the students were struggling on in their groups. This was a prerequisite activity to review mathematics that was needed to work on the next activity.

Sue then presented the students with a challenging Pepsi problem. She showed the students a 2 liter bottle and a 20 ounce bottle of Pepsi. She told the students that the 2 liter costs $1.59 and the 20 ounce cost $.89. She asked the students, "Which one is the better buy?" She also told them to figure the price per ounce and the price per quart. She then had the students get back into their groups and work on the problems. Students started using multiple strategies to attempt to solve the problems. Sue changed the focus of the activity to be on how the students should set up the problem based on how the students were reacting to the activity. Sue told the students, "I want you to know the process and not so much the calculation." She asked the students to go back to their desk and conducted a whole class representation of the problem on the white board using one of the groups work as an example to emphasize same unit comparisons. Sue then grabbed three different colored dry erase markers and made a representation of the students work in red.

Sue used a series of questions to guide students through the process of setting up a proportion. She told me in an interview, "I just keep asking questions because I think you get kids to keep thinking about what do I do next?" As she is asking questions and the students are responding she uses blue and green markers to add reactions and emphasize important points. After Sue walked the students through setting the proportion up correctly she had the students get back into groups and figure the price per ounce and price per quart for each example. She then used representations and guiding questions with the students to solve the problems. Group work helped Sue decide how students understood the concept and procedure and determined how she would proceed with the lesson.

Sue then extended the problem to test student understanding and provided practice with the skill. She changed the price of one of the items and asked, “What if I changed the price of the 20 ounce to $.50, what would be the better deal?” After students had figured out that problem she asked, “Let’s do another comparison. If I bought a gallon of milk for $4.28 what would it be per quart? You can do this in your head.” She made comparisons to other products and discussed other ways that people make purchasing decisions other than price. Sue started this Pepsi problem lesson by creating a problem. She said in an interview,

> I like to start out with a question and get them to wonder about something. Let them try it out and come back to how we solve this together and go through what’s the mathematical way to figure this out.

Sue created activities for students to interact and explore the mathematics through extending the problem in ways that showed the practical value of and multiple ways to use the procedure. At the end of the Pepsi problem, Sue took advantage of a mathematical teaching opportunity. A student figured the price per quart that had an answer of $2.86 65oz. x .044 = $2.86). Sue had just did the problem using proportions (.89 x 65 oz = 20 oz. x b) to get an answer of $2.89. Sue jumped at the opportunity to explain why there was a difference of three cents that was not a part of the lesson. Sue told the students, “With small numbers you may need to use the hundredths place to get small differences in an answer.” Then she showed the students that the three cent difference was the difference between the student's rounded price per ounce of .044 and the actual price per ounce of .0445 which was accounted for in the proportion method. She showed the differences by performing the procedures on the board.

Communication between students, other students, and the teacher were continuously occurring in my observations. Notice in the next teaching episode how Sue used communication to explain the mathematics in ways that built deeper understanding for the students. She asked the class, "Why did I give you an assignment with perimeter and circumference?" The class did not respond so she asked, "So tell me what perimeter is?

John: The area around.

Mrs. Johnson: We're not going to use the word area. We just went through it. What's the proper way to give the definition?

John: The measure around it.

Mrs. Johnson: Okay, the measure around it and what? What are we measuring? We are doing perimeter. I want a good definition of perimeter. Tom gave you a good definition the other day and gave you a good definition today. Your definition was not a good definition because you put one word in there that makes it very confusing. I want a correct definition. So tell me again what's the definition?

John: Outside of an object.

Mrs. Johnson: Okay, the outside of an object and how do you measure it? What unit do you measure?

John: Centimeters

Mrs. Johnson: Well we want centimeters, but what are we measuring? Okay, when we are measuring the outside we are only measuring the one dimension. What is it? What's the one thing we're measuring? (pause) Sandy?

Sandy: Length

Mrs. Johnson: All we're measuring is length. So what did John say when he first gave me the definition of perimeter that made it confusing? Tammy?

Tammy: Area

Mrs. Johnson: Okay, we don't want to say the perimeter is the area around something because we are then confusing the words perimeter and area. We need to either say it is the distance around something or the length around something because that helps us to remember that we are just measuring length. Okay, and back to the second question that I had for you, John, Why did I give you this assignment on perimeter and circumference together? Why did I put those two lessons together? There is a relationship between them. What's their relationship?

John: They are the distance around something.

Mrs. Johnson: Okay, they are the distance around something. So what's the difference between perimeter and circumference?

John: The distance around a square and a circle.

Mrs. Johnson: Is this a square?

John: No.

Mrs. Johnson: Okay, it's a polygon. Perimeter is the distance around a polygon and circumference is the distance around a circle. We reviewed how to find perimeter. How do we find circumference?

Tim: By taking radius times 3.14.

Mrs. Johnson: And 3.14 is really what?

Tim: Pi.

Mrs. Johnson: And do I take the radius times pi or the diameter times pi?

Tim: Diameter.

Mrs. Johnson: Okay, so what if I only know the radius?

Tim: Take radius times two.

Other students in the class who were not called were able to compare their own understandings to the one's being discussed by the teacher and other students. Students had to pay attention because Sue randomly selected them to respond to questions. Sue created situations that asked students to express their personal understandings of the mathematics so she could help them develop deeper and more complete understandings.

I will present three more teaching episodes to describe Sue's teaching. Notice how Sue guided her students' understandings of mathematical reasoning. For example, when

Sue was trying to help students understand decimal points to percentages in a lesson on proportions she used estimation as a logical number sense technique. The problem was 48 is what percentage of 120?

> Student: four percent
>
> Mrs. Johnson: Logic will tell us that that's not right. Why would logic tell us that that's not right?
>
> A student: You can check it by multiplying 120 x .04 to see if it is close to 48.
>
> Another student: It should be close to fifty percent because 48 is close to half of 120.
>
> Mrs. Johnson: One half of 120 equals sixty. (She writes on the board, 50% = 60) 48 is not far off from 60 is it. So it has to be a little bit less than ½ but is it going to be 4 percent? Four percent is not very much because I know what ½ is. So the answer should be somewhere related to that.
>
> Another student: (using his calculator) .4
>
> Mrs. Johnson: What does that mean? Is the answer .4 percent, 4 percent or 40 percent?

The calculator did the procedural part of the problem but Sue wanted the student to make sense of the result to understand the solution. Sue then taught about the equivalency rules of the different ways to represent a number. For instance, 40%, .40, .4, and 40/100 are all representations of 40% furthering student's logical understandings and sense making of numbers. Sue wanted her students to understand what they were doing and why they were doing it and not just compute the answer. Sue exclaimed in an interview, "Good number sense and logical answers go hand in hand."

In an interview Sue said,

> Over the years we've taught algorithms and formulas but we didn't have an understanding of it. What am I really doing here? Why are we doing that? Why does that happen? It's like the use of zero in two digit multiplication. Instead of just telling the students to just put a zero there on the second line it is important to show why do you put that there.

In another teaching episode, Sue was working with students on the area of a triangle. She wanted the students to know what the formula was (base times height divided by two) and why it works. Students were confused with the term “formula” and how to write it. Sue wrote on the board as she demonstrated and explained how to write a formula. She reviewed algebraic notations with the students and used the example of “S” to represent a side and wrote formulas for different types of triangles. Sue said, “It’s a recipe that works all the time.” as she wrote on the board,

	<u>Perimeter</u>
	S + S + S
Triangle	3S “For an equilateral triangle”
	S1 + S2 + S3 “For a scalene triangle”

The students were confused about how to write a formula for the area of a triangle asking if they should multiply or divide by two. Sue took one of the triangles a student made in yesterday’s class and put it on the board. She asked a student to come up and put “b” for the base and “h” for the height at the appropriate positions and had other students explain why the positioning was correct using precise terminology. Sue took another triangle and added it to the first one to create a parallelogram.

Mrs. Johnson: What is the new shape?

Students: parallelogram.

Mrs. Johnson: What is the area of a parallelogram?

Students: Base times height.

Mrs. Johnson: The base was seventeen and the height was fifteen so the area of the parallelogram was 255. So how can I prove that formula? So what if we take 17 x 15 and divide it by two?

A student: 127.5

Mrs. Johnson: And what if I take 17 x 15 and multiply by two?

A student: 510

Mrs. Johnson: So that is why we divide by two instead of multiply by two.

Sue emphasized that there were two triangles in a parallelogram so they should divide by two. She wanted the students to understand why the mathematics worked as well as being able to execute the proper procedure of the formula. Sue told me later in an interview that when she was learning the formula she couldn't remember whether to multiply or divide until she was given a proof. "When I learned why you divide by two it helped me remember and understand the procedure and, therefore; it meant more to me than just a math formula."

Sue started the teaching of each new concept or topic by having the students attempt to solve a problem or puzzling situation without giving directions to solve it. Sue said in an interview,

> I give them (students) a situation or question and ask them to figure it out. Get them to wonder about something and let them try it out for a little bit and then come back to how we solve this together and go through the mathematical way of figuring this out.

For example, Sue challenged the students to find the area of a trapezoid in an introductory lesson. The trapezoid had a rectangle and two triangles. Sue drew the trapezoid on the board that included measurements. She included measurements for base one and base two, the height and for the slants. As she wrote the number for the slants she said, "I'm trying to stump you." What she was really doing was testing students' misconceptions about area. The class was able to find area of the trapezoid using skills they had learned previously (area of a triangle and area of a rectangle) and some collaborations with each other and the teacher's guidance. Sue asked the students "Do you have to divide by two" when the students were figuring the area of the two equal triangles. Sue introduced the formula for a trapezoid and asked the students to try it out on the example to see if it works. She then asked, “Is ½ (b1 + b2) x h the same as (b1 + b2) x h/2?” Sue continued to extend the problem and asked the students if the formula would work for another trapezoid that had a rectangle and one triangle. She showed the students some examples of the distributive property to make connections between the formula and its modification to the extended problems.

Sue asked students to make conjectures and test them to see if they could provide a proof for a problem. For example, Sue wanted her students to learn the formula for circumference (c = d x pi or c / d = pi) of a circle in a lesson. She wanted the students to make sense of the formula instead of just memorizing it. She also wanted students to make conjectures and test to determine if they could prove the conjecture. The students conjectured three possibilities:

1. r x r x pi / 2
2. r x pi / 2
3. r x 2 x pi

Sue told the students not to use their resource book to find the answer but to explore objects (circles) to see if they could prove one of the formulas to be correct. In groups, the students measured circular items around the room and tried the conjectures out. The class was able to eliminate two of the conjectures and modified the other to come to the correct formula that always works through interactive exploration and Sue’s guidance. Misconceptions were identified and corrected and deeper understanding was achieved through these explorative experiences. For example, Sue showed students that 2r is different than r2 (squared) as well as other combinations of the formula such as 2r x pi = c.

Sue used representations in every lesson I observed in all grade levels. Sue told me in an interview,

> I guess I like to draw a picture a lot of times because I think that helps them see it. It gives them that model or something so that they can make sense of it logically. I like to draw pictures to kind of show a whole process and have them practice on boards or get together in groups or whatever to kind of re-discuss to get a little bit more familiar in their eyes.

Some of Sue's representations were simple. For example, a boy was struggling on a coin toss problem with ratios. Sue said, "You look confused, I'm going to draw this picture." She drew seven circles to represent the times the coin was tossed. She then put an "h" in four of the circles to represent how many were heads. This visual representation provided the student with the part of the problem he was missing without telling him. The student now realized that there were three variables to the problem and could make multiple ratios. Sue used representations to show students relationships between mathematics, mathematical language, and verbal thinking about the mathematics.

Sue used representations to externally examine students' thinking about the mathematics they were doing. For example, students were given problems that had extra steps to finding which labels and numbers to use in setting up and executing proportions.

> Mrs. Johnson: They didn't give you the word. You're going to have to figure out how to set up the proportion.

She then went over a practice problem to model the process students will need to execute to solve the problem. She continued,

> Mrs. Johnson: There are 2 face up for every face down card. If 6 cards are face up how many are face down? So, how can we set that up as ratios and proportions, Ted? First of all what is going to be our words?

Ted: Face down cards and face up cards.

Mrs. Johnson: Okay, face down and face up.

Sue then draws this representation on the board:

$$\frac{Fd}{Fu}$$

Mrs. Johnson: I don't try to write all the words. Remember, I said that mathematicians don't like to write so I'm just going to use abbreviations. So, what do I know?

Ted: There is two face down cards for every face up card.

Mrs. Johnson: So, what do I put here?

Notice that Sue didn't just put the numbers into the representation. She asked the student to tell her what number goes where. The student answered her and she added to the representation.

$$\frac{Fd}{Fu} = \frac{2}{1}$$

Mrs. Johnson: Okay, and then it says what?

Ted: That there's every, for every sixth card, for every six cards up.

Mrs. Johnson: Does that go here? (And added to the representation)

$$\frac{Fd}{Fu} = \frac{2}{1} = 6$$

Ted: Yes! So, so you got to figure out how you got from one to six. So, you have 1 x 6 = 6. So then you take 2 x 6 to get the other.

Sue added to the representation as the student was verbally explaining:

$$\overset{X6}{\frac{Fd}{Fu}} = \underset{X6}{\frac{2}{1}} = \frac{12}{6}$$

Mrs. Johnson: Okay, so how can we check our answer, John?

John: You can take 2 x 6 and 12 x 1.

Mrs. Johnson: So 2 x 6 is 12 and 1 x 12 is 12 so that confirms that our answer is correct. Okay, that one was fairly simple, but again, we put our labels so that we know what is what. I want you to try two or three on your own. Now, I'm going to caution you because sometimes were talking about all cards and not face up or face down and that's when it gets to be a little confusing.

After a period of individual practice on a problem Sue brought the class back to a whole class activity where she had a student write her problem and answer on the board.

Mrs. Johnson: Okay, I don't want just an answer, I want to know how you got there. I want you to do just like I did. I want to see your labels, your numbers, and your computation.

A girl put her answer on the board:

$$\frac{U}{D} = \frac{3}{5} = \frac{12}{20} = 20 \text{ cards}$$

Mrs. Johnson: Does everyone agree with what she has there?

Three students agree.

Student A: I think the labels are wrong.

Student B: The labels are okay.

The student that wrote the answer on the board goes back up and starts changing variables as the class disagrees with her actions.

Mrs. Johnson: Now you're making her paranoid.

Class: Laughs

Mrs. Johnson: Let's look at the labels. Three out of every five cards are face up.

This brings plenty of conversation among the students. Sue then redirects the students to the labels.

Mrs. Johnson: It says 3 out of 5 are face up. So, 3 are face up and that (5) should be total. So, it didn't say 3 up for every 5 down did it. It said 3 out of every 5. So 5 was total. So that is where we sometimes get into trouble because it was just like the problem with rain yesterday. It said 2 out of every 5 days it rains, how many times does it rain in April? We had to know how many days in April there were.

The class worked on another problem and asked the teacher if they could draw a picture.

Mrs. Johnson: I assumed that you would be able to do it on your own. Okay, Fred go up there and the rest of you need to look at what he is doing to see if you agree. Fred, if you can explain what you're doing along the way that will help. Labels come in real handy here because they can help you see your mistake.

Notice how Sue is allowing the students to explore the mathematics as she guides their focus and direction.

The boy wrote on the board:

$$\frac{Fu}{Fd} = \overset{\times 3}{\underset{\times 3}{\frac{2}{5}}} = \frac{6}{15}$$

The class discussed the answer. Sue interrupted the conversation.

Mrs. Johnson: The question was how many face up cards are there? He is right to say that there are 6 face up cards. There might be a better way to explain that. Michael, how did you do it?

Sue knew that Michael knew a part of the problem the class was exploring and wanted him to show the class his discovery.

Michael: I had 2 face up cards for every 5 face down cards. So I figured that I had seven. And kept the two face up cards. Two plus five is seven.

Mrs. Johnson: Okay, show that on the board.

Other students: What? That will even be more confusing.

Michael went up to the board and changed the representation:

$$\frac{Fu}{Fd} = \overset{\times 3}{\underset{\times 3}{\frac{2}{7}}} = \frac{6}{15}$$

Mrs. Johnson: If you change that to seven, what else are you going to have to change?

The student didn't know what else would change.

Mrs. Johnson: I agree that it would be okay to change that to seven and I know where you got the seven but I'm not sure that everyone else does because if you're going to change that seven you have to change something else.

Another student: You have to change the top, don't you?

Some of the other students agree.

Mrs. Johnson: Well that's true in some situations but over here there is a reason why he changed that five to a seven.

Student: He added 5 to 2.

Mrs. Johnson: And why did he add the 5 to the 2? Michael, do you know why you added the 5 to the 2?

Michael: Because you can't have.. you have 2 up and.. 5 down for seven.

Mrs. Johnson: So you have seven what? So, you have seven face down cards you have seven what?

Michael: Total

Mrs. Johnson: The reason he wanted to do that is because the question asked if you have 21 cards in all how many cards would be face up. So this is one example of when we first started talking about ratios. We talked about how, for example, when I said what's the ratio of boys to girls in our class? The ratio was 6 to 3. But what if I said what is the ratio of boys to all the kids in class? Then it was 6 to 9.

Sue drew a representation on the board that simulated what she was saying.

$$\frac{B}{G} = \frac{6}{3} \qquad \frac{B}{All} = \frac{6}{9}$$

She used what students already knew to help them understand the initial problem they were confused about.

Mrs. Johnson: That is kind of what we had to do here. Up to down was 2 to 5 but you needed to find the ratio of up to all cards so that you could answer the full question. That is where trickiness comes in reading and solving proportion problems.

Sue erased parts of the original representation and put in the correct responses. Sue used visual representation to see what mathematics was being discussed and to help students make connections between the procedure and the concept to better understand what they were doing and why they were doing it.

The final representation to the initial problem was:

$$\frac{Fu}{Total} = \overset{X3}{\underset{X3}{\frac{2}{7}}} = \frac{6}{21}$$

In this problem the answer was 6 face up cards and the students got the answer right but they did not have the correct procedure or concept of the question. This was identified through the process of interacting with the mathematics through representation and communication. Sue asked her students to show their work on all assignments and class activities and would occasionally have her students work problems on individual whiteboards if she was interested in seeing how students understood a process. Sue said in an interview when discussing the individual whiteboards, "You find out what misconceptions students have and who understands it and who doesn't understand it. Where I see where they are making their mistake."

Sue finished each math lesson with an assignment and began the next lesson picking up their assignment and provided feedback on students' performance. Frequently, Sue changed the assignment for the class based on what she felt the students were ready for and by how far she was able to progress through the lesson. She wrote feedback on each student's paper and discussed and reviewed specific mathematical problems or concepts that students were having trouble with before she started her next lesson. Occasionally, she reviewed pre-requisite skills students would need to successfully complete that day's tasks before the lesson started.

Key issues.

Dialogue. Sue used group work to create dialogue in every lesson I observed. Students had to explain their thinking and reasoning to each other and discuss their perspectives on the mathematics. Students comfortably agreed or disagreed with each others processes or concepts. Then they discussed the discrepancies with each other and explained how they did it. Sue felt that this process helped students find their own mistakes and correct incomplete understandings and procedures. This process not only helped students understand the mathematics better through communication it also informed Sue about what she needed to do next to help students develop mathematical skill both conceptually and procedurally. In the circumference lesson, Sue asked students to explore circular items in the classroom to see if they could figure out what the formula was. She did not offer any solutions so the students had to talk about their ideas and understandings to solve the problem. Sue used information from the dialogue of each group to assess their understanding and procedural skill to determine how the lesson progressed.

She frequently interrupted the dialogue during group work to redirect students and highlight a group's discovery. Sue frequently used work that students were doing in their groups as models to explore the concept or procedure she was teaching. Sue noticed from the dialogue while working on the Pepsi problem students needed instruction about equal measurements in cross multiplying and proportion problems. She used the work of one group to teach the concept and procedure. The group had set up the proportion using quarts for one and ounces for the other. Sue guided students to find the error and developed their thinking through whole class dialogue that included questioning and representation. Sue told me that for students to really understand something they must be able to tell someone else how they did it and why. Sue used dialogue throughout her mathematics lessons. She wanted students to use the proper mathematical language in the discussion with students as shown in the circumference lesson. Dialogue with the student and the class helped the students understand better the difference between area and perimeter.

Making logical sense of the mathematics. Sue made a conscious effort to have students make logical sense of the mathematics they were performing in every lesson I observed. She created situations, problems and extensions to problems, and asked specific questions to explore students' sense making and logic of the mathematical skill being taught. Sue showed students ways to test their logic, conjecture or answer. She wanted her students to understand what they were doing, why they were doing it and how they could transfer the skill to a new situation. In the 48 is what % of 120 problem, Sue wanted to know how students understood the calculators answer of .4. She allowed her students to use calculators as a tool but expected them to do the thinking and understanding of the problem. She expected students to explain how they got the answer whether they used a calculator or not. Her questioning strategies and communication techniques helped students understand why the mathematics worked.

Sue was always looking to develop deeper understandings of concepts and procedures. She asked students to explain their reasoning and then created ways for them to develop mathematical proofs for their thinking. Sue used student work and asked questions to show the procedure to area of a triangle when the students were not sure if they divide or multiply by two in finding the area in the triangle problem. She showed how two triangles make a parallelogram and showed a proof of the logic that determined the answer to their question. She wanted the students to understand why the mathematics works.

Representation to "see it." Sue used representations on the whiteboard or overhead projector multiple times throughout every lesson. She also had students write their own representations or interact with the one she presented. Sue's representations helped students make connections between what was being said about the mathematics to the mathematical language or a visual depiction of the logical process that was being implemented. These visual representations showed students what was happening mathematically and allowed them to interact with the process to develop deeper understandings about concepts, procedures, proofs, and provided clues about future conjectures or what to do next.

Sue used representations to see what students knew and what they did not know about a concept or procedure. Sue asked students to make representations and interact with others to externalize what students were internally thinking. In the proportion problem solving lesson, Sue used representation to discover that the students had gotten the right answer but did not have the proper concept or procedure to the problem.

Exploration and extensions. Sue wanted students to explore mathematical ideas and make conjectures about how to solve problems. She frequently asked students to explore possible solutions to problems without guidance in my observations. She wanted students to develop experience with the problems before she presented the mathematical solution hoping that the students were able to create the proper conjecture or have experiences that contribute to the development of the correct proof. In the Trapezoid problem, Sue challenged the student to find the area. She gave the trapezoid and the lengths to the students that included the slants. Through exploration the students developed mathematical experience to contribute to the understanding of the formula and possible variations.

Problems were extended frequently in Sue's classroom. Extensions to problems helped students see how the skill could be used practically in other situations. She also extended problems when opportunities to understand other mathematical concepts and procedures arose. In the Pepsi problem, Sue changed the prices of the different products to allow students to explore which one was the better buy. Then she switched one of the product's type or size. Sue showed why two different processes of determining the value of the Pepsi products were different by three cents.

Feedback guides instruction. Sue Johnson knew what mathematics she wanted to teach in all of my observations. How she taught a lesson depended on how the students were reacting to the verbal, visual, and procedural tasks that developed. "I guess

I divert a lot," she said. Sue diverted, extended, adapted, and changed her plan frequently and comfortably at an instant. Sue designed specific activities and questions to see and hear what students were thinking about the mathematics and what they understood about what she was teaching. Sue gave students an assignment each day to practice the computational skill that was taught. She decided which problems she wanted the students to do at the end of the lesson. On multiple occasions Sue changed the order and amount of questions based on how far the students progressed in the lesson. She told me later that some of the questions were not appropriate to assign because she had not taught the necessary skills yet.

Sue taught her lessons spontaneously based on the feedback she received as the lesson progressed. Group work, communication, exploration, homework, assignments and representations all contributed feedback to Sue about students' understandings and acquired skills. These pedagogical activities guided Sue to determine what action to take next as the lesson was in progress. In the perimeter lesson, Sue realized that some students did not have a deep understanding of the difference between perimeter and area. She instantaneously shifted the lesson to a detailed whole class discussion on the proper conceptualization of the differences. These unplanned occurrences happened frequently in my observations. She explained,

> I always say that when I write my lesson plans for the next week on Friday by Tuesday we are going to be off because I really see that they are not getting something and I have to take the time and go back over it. If they are not getting it then I slow down and take more time or try to find a different way to explain it or maybe do more practice problems.

Sue had a plan about how she thought each lesson would go but was flexible to adapt to the needs of her students and was comfortable to take the lesson where she felt the students needed it the most. “I feel that I get to where I want to go better when I work more spontaneously,” she said.

Participant: Kathy Freeman

Context. The drive to Kathy Freeman’s school was filled with traffic and stoplights. The school was located two miles south of the downtown area in a large city of 250,000 residents. Traffic was heavy and took approximately 15 minutes to travel about 5 miles. As I got closer to the school I traveled from commercial to residential areas. The school was nestled into a neighborhood with mature homes that had been constructed over 30 years ago. Each house had its own architectural design with mature trees on most properties. The houses had small to moderate sized yards that were generally well maintained. The brick school was located on the west side of a busy street. A paved square parking lot was located right off the street to the right side of the front of the building. The end of the lot was attached to a vertical sidewalk that led to the front door. There were 20 parking spaces available and were mostly full when I arrived. Less than 100 feet from the building was a paved semi-circular driveway that contained some additional parking spaces and some room for cars to drive through. This area was constructed so parents could drive off the busy street to pick up and drop off their children without parking. Between the driveway and the street were two cement spaces for the buses to pull off the busy street to load and unload students. In the middle of these two bus stop indentions was a stop light that was specifically designed to help students cross the busy narrow street. It only changed from green to red when someone activated the button on the side of the pole that housed the lights. Residential houses lined the front and sides of the building.

The back of the building, where the playground is located, had plenty of land. The playground right out the back doors is located on gravel and had colorful plastic gym equipment with ropes and bridges. Another area had climbing apparatus equipment. There was a large sand pit where students could play under a large shade tree. To the right of the playground were two basketball rims on cement with string nets that were torn. Beside the basketball area was two portable classrooms. Directly below the playground was a stairway that led to a lower level playing field the size of a regulation football field. There were three sets of soccer goals within the football field and a baseball wire fence backstop on the corner of the field. The playground and the playing fields were surrounded by chain link fences that were bowed over in places.

The main entrance to the school was located directly behind the stop light in front of the building. There were six steps up to the main walk way into the building and had two black painted benches on each side of the side walk. There were many mature trees in the front of the building that yielded a mostly shaded area. There was a flag pole on one side of the sidewalk. It displayed a US Flag with a State flag below it. On the other side were four flag poles surrounding a circular brick patio that displayed wind flags of the school's colors of yellow and blue. There were small areas of flowers, shrubs, and plants to bring natural color and beauty to the area. There were six glass doors at the front of the building to enter the school. As I entered the building I came directly to a table against a wall with a small enclosure into the hallway. On the left side before the table was a door to the office with windows so that office personnel could identify who was entering the building. The table was manned by a school official who checked me into the school. I signed in and out of the school each day on a computer. I was given a badge that was placed over my neck that stated "visitor" and had a number on it. I was told to wear the badge at all times while in the building.

The office called Kathy to come and take me to her classroom. The building was rectangular shaped and had hallways left and right and up and down. As I walked down the hallway to the right there were many postings of students' work on the walls. One side of the hallway had lockers and the other side had windows, bulletin boards and hangers for students' coats. Kathy's room was on the right side of the hallway. As I walked into the room I noticed that the walls and bulletin boards on each side of the room were filled with mathematical information. In the front of the classroom were two bulletin boards. One was the math calendar and the other was a math word wall. Also, on the front wall was a hundreds chart, money charts, an interactive clock, the numbers one through ten individually taped to the wall, and a list of "math shining stars." The one side of the room was cabinets, counters, and a wooden closet where Kathy stored all of her mathematics games and manipulatives. The large closet that was normally used for coats and large boxes was packed with hundreds of games, manipulatives, and math materials, on 5 shelves that have been collected over the last 30 plus years. Kathy told me that she was able to purchase these materials from funds she received from the monetary stipend she earned from her Presidential Award in mathematics. Kathy organized the shelves and was able to find materials instantly when needed. On the other side of the classroom were windows and felt walls where Kathy had charts and portable chalkboards. The back of the classroom was the teacher's desk, a computer center and a semi-circular table where the teacher could work with small groups or individually with students.

This first grade classroom was very colorful and busy. There were many boxes, books, folders, and plants all around the classroom. There were five large tables that housed four students, two to a side. There were square containers on each table that housed the students' supplies. The tables and unattached chairs were appropriately sized for first graders. The tables were arranged vertically in the back two-thirds of the classroom to give plenty of space in the front for the whole class to sit comfortably in front of a large whiteboard. Kathy had a female student teacher that was in the class every day for the whole day. She mostly graded papers, made preparations for future assignments, conducted office and classroom management duties, and worked with students on a one to one basis. She did not teach any of the mathematics classes when I was observing. There were 12 boys and 10 girls in the classroom. Four students left the classroom during mathematics everyday that I observed. One was a special education student and the other three left when a teacher called on the telephone, usually half way through each observation. The phone rang two times and no one answered it and the three students got up and left class without being told. Kathy told me in an interview, "Those kids have struggled all year and I worked it out in the last two months so it gives them the extra help they need because they weren't being successful in my classroom and I wasn't able to give them the extra help they needed."

Teacher. Kathy comes from a family of educators. Her father was a professor at the local university. She was the oldest child in the family with three younger brothers. "I taught them so I figured that I should teach school. Education is in our family and something I wanted to do. I really like watching kids learn, especially first graders." Kathy taught her first 3 years for another school district and the last 34 years at her current school. She has always taught first grade or a combination of first and second grade and has always loved mathematics. "I've always loved math as long as I can remember, even in elementary school I loved math. So it's just something I enjoy." Kathy explained about her public school experiences in an interview,

> Math was just workbooks and pages. It was computation. Nothing was hands-on. I never remember math being hands-on when I was in school. Nothing to visually help you understand what you were doing. I could do the problems, but I didn't understand what it was that I was doing or why it worked.

Kathy has a bachelor's degree in elementary education. She also has a master's degree in elementary education. She wanted her master's to focus on mathematics but said, "I took everything I could in math but we really didn't have any program to kind of go in that direction." Kathy took a mathematics methods class in her undergraduate studies. She talked about her experience in an interview,

> It was the very first time they taught it. I think they've since required that now, but at that time they didn't. I just took it because I like math. My math class was with Mrs. Cavett. She probably turned me on to teaching math as much as anybody. She used manipulatives to teach math and to get kids to have an understanding. I really enjoyed that class and that really got me motivated.

I asked Kathy about her mathematics background in an interview. She exclaimed, "It's surprising that I don't have a lot of mathematics background. I regret that I didn't take as much math in high school as I could have." Kathy filled the gap with professional development activities and research on mathematics. "I've attended a lot of NCTM conferences all over the country and I take classes when ever I can," she said. Kathy also reads professional books about teaching mathematics. She added,

> I love to read and I love math. I read math kinds of books to get information on what kinds of games work, what kinds of activities work for kids. I like to find out what is going on in their heads so I try to read books that will give me information on how kids think.

Kathy is a member of numerous professional organizations that include: NCTM, NATM, NEA, NSEA, and CPAM.

Kathy has taught in an elementary classroom for the last 37 years. She has also taught a Peers Academy for the local university. This was a two week summer workshop in mathematics for practicing teachers. Kathy focused on using manipulatives to teach mathematics. She has taught a methods course at the university for pre-service teachers and has been a speaker at state and regional mathematics conferences. Kathy is a winner of the Presidential Award in Mathematics in 1996.

Survey. Kathy Freeman took the CKT-M (content knowledge for teaching mathematics) instrument from the University of Michigan's learning mathematics for teaching project. Table 3 shows how Kathy scored. Her IRT scores shows that she performed about

Table 3

CKT-M Scores for Kathy Freeman

Construct	IRT Score
Numbers & Operations	0.900
Patterns, Functions, & Algebra	1.243
Knowledge of Students & Content	0.884

one standard deviation above the average teacher in all three constructs. Kathy missed 3 questions that involved fractions. These questions asked what the "best" interpretation was or "most likely" to be the reason. I noticed that Kathy struggled with how to answer these questions when we discussed the results in an interview. Her responses to me included, "They both work" and "They all are reasonable mistakes that students make." Kathy's teaching valued multiple ways at looking at solutions to problems from students and struggled trying to value those ideas.

Kathy's teaching. In this section I integrated observations and interviews in my description of Kathy Freeman's teaching. Also, within the writing of her teaching I used examples during multiple lessons to show a more complete picture of her teaching. I observed nine consecutive math lessons of Kathy's 1st grade classroom. I conducted three semi-structured interviews one-on-one with Kathy throughout the nine days of observations during her planning time following mathematics class. The unit that Kathy taught during my observations was on addition and subtraction for both single and two digit problems. Some of the lessons focused on fact families, adding three single digit numbers, missing addends, and using strategies to add and subtract numbers. A major focus of all of her lessons was on number sense and developing basic facts for adding and subtracting numbers under twenty. Her daily mathematics class was 1 hour and 10 minutes and divided into a series of mathematical tasks of about 10 to 15 minutes each which helped her students' first grade attention spans. I presented teaching episodes that highlighted each of those segments of her teaching.

The designated time for mathematics in Kathy's classroom was in the afternoon following recess. She also conducted "Calendar Math" in the morning for about 10 to 15 minutes and some activities at lunch time interacting with students and numbers. Kathy did not like that mathematics class was scheduled in the afternoon. She explained in a discussion with me, "I won't do that again. This year they had afternoon math to meet multiple schedules." At this time of day it was hard for students to focus on and think about mathematics. Kathy told me, “This year I have a lot of behaviors so I have a lot of things that are interfering with our learning. I ignore some distractions to get something done.” There were three students that had behavioral problems during my observations. At various times I saw them talk back to the teacher, defy the teacher’s directions, argue with other students and ignore the lesson all together. On three days of my observations the class included a para-educator to help with behavior problems. After mathematics, the students prepared to go to “specials” that included physical education, music, and art depending on which day it was. Students cleaned the tables, put their chairs upside down on the table, and went to their lockers in the hall to get their backpacks and put them on top of their chair to go home right after specials. Kathy’s afternoon mathematics class was scheduled for one hour and ten minutes. She divided the time using multiple activities that included: games, timed tests, problem of the day, discussions, board work, worksheets, and summary. Kathy used most of these activities every day in math class depending on how the activities went in the allotted time. She also changed the sequence of the activities on different days and during lessons based on how students were reacting to the lesson.

I arrived a few minutes before mathematics class started and set up in the back of the room beside the teacher’s desk while the students were at recess. I purposely did not interact with the teacher or the students throughout each lesson because I wanted the experience to be authentic and unobtrusive. The students got a drink of water in the hallway on their way into class. They came into the classroom actively talking and moving around and ended up at their chairs. When everyone was seated, Kathy instructed them to find out which group they were in and to start the games portion of math class. The students and teacher were interactively playing games, manipulating objects, and solving problems during mathematics time, mostly away from their original chairs.

The first activity I saw was games. Games was the first mathematics activity in 6 of the 9 lessons I observed. Games took place for 10 to 15 minutes. Students worked in groups of 2 or 3 that was posted on the bulletin board. These groups changed twice during my observations. The game that each group played changed every day. Kathy carefully selected the groups that were usually of the same gender. Students went up to the board to find their group. Then they went to the games box to get materials for that game and went to a table or the floor to play their game. There were 9 or 10 different groups playing different games at the same time across the room. Kathy changed games after everyone had a chance to play each game. Most of the games that the students were playing were base 10 games. There was a 10 frame game, 10's go fish and other card games, road hog and other board games where students were to count and add and then apply the result to the specific game. Kathy made two of the games called wild card and Dino-mite. These games were also counting games where students interacted with numbers and each other. Kathy talked about why she decided to use games in her teaching during an interview,

> How am I going to teach the basic facts? I'm hearing out there that research says just doing a timed test every day doesn't teach them their basic facts. So what can I do that is fun and that's kind of where games come in. They don't even know they are learning their basic facts. They just like the games. I started researching games and some of them are good and some of them are not good in my eyes. I don't just choose a game and say let's play. It's got to have a purpose.

The noise level went up during the games as students would discuss the counting, adding, and moving objects within the game with each other. Kathy went around checking to see if the students were playing the games correctly and usually would sit with a group that she could work with specifically. She also took time to work with a specific child or two during this time on their individual mathematical needs. Occasionally the students would be off-task or play the game incorrectly but were usually involved in socially counting and adding experiences with numbers. Kathy would occasionally allow a few students to play other games that were not related to base ten or basic facts development such as pentominoes, tangrams, Cuisenaire, and other math puzzle games. Kathy talked about using certain manipulatives in an interview.

> I don't use as many manipulatives as I did before because we're more focused on certain objectives with students so I don't get to use the pentominoes and tangrams in my classroom much. But I believe they have an important place. Children don't work puzzles much any more. With pentominoes and tangrams they have to turn them and flip them and get them to fit into the space and that is really important.

Kathy used manipulatives and games in every lesson I observed. She explained reasons why manipulatives are an important part of her teaching of mathematics in an interview,

> I think that manipulatives are a great way to have them experience what mathematics is. I'm a real constructivist. I prefer to teach math where they construct their own learning. Where you give them the tools they need and let them just explore and try to solve a problem you've given them and let them just construct that learning and get a basic understanding. Third grade teachers think they (students) don't need those things and they really do. Most kids learn that way. They'll learn it and it will stay with them if they use their hands and their eyes to do it.

Kathy used timers, music, count downs, and songs to signal the end of games and other math sections. One timer would be used for the activity and another for the transition to the next activity. She used these strategies to stay on time and to help students transition between activities efficiently. She also used music to control the noise level in the classroom.

Kathy conducted timed tests in 4 of the lessons that I observed. The timed test was a part of the districts curriculum on mastery of the basic facts. Students had 3 minutes to compute as many problems as they could on a sheet of paper. The students sat at their assigned seats for this activity. There were 20 problems to complete in the allotted time. Kathy handed out the sheet of paper and each student would put their name on it and turn it over showing they were ready to start. Kathy reminded the students that they needed to get 17 or 18 correctly done to get a good score and that, “We don’t erase a wrong answer, just cross it off. Remember, mathematicians check their work, yes they do.” When Kathy said start she started the timer and the students turned over their paper and rushed to finish it as fast as they could. This timed test was on subtraction basic facts. Later they did one and two digit addition facts tests that had varying allotted times to complete the 20 problems. Some students finished within one minute. They turned their paper over and drew pictures on the back quietly until the rest of the class was done. Kathy collected the papers in alphabetical order when the timer went off. She had to score each student’s paper and record it on a computer spreadsheet for the district and to send home to parents showing how their child was performing on memorizing their basic facts.

Kathy told me that she has to reteach and retest students if they did not pass a mathematics chapter test with a score of 80% or better. “I have a volunteer in the community help me, a retired teacher.” Kathy told me in an interview that she wrote down the objective and collected the materials for the volunteer to reteach and retest students. I did not see this service being conducted during my observations.

The problem of the day was conducted in every lesson except on the day that students took the unit test. Students were called to get a board, marker, and towel and meet on the floor in the front of the class. Kathy counted to twenty as the students hurried to get ready for the next activity. The problem of the day was written on a large presentation board made by the curriculum's publisher. One problem was on each page and it was flipped over for the next day's problem. The problem of the day corresponded with the worksheet and the day's lesson in the curriculum. The problem of the day was a word problem in all of my observations. Kathy read the word problem and the students attempted to solve the problem by writing on their white boards with dry erase markers. When the students had a solution to the problem they were suppose to cover it up or turn it over. Kathy said, "One, two, three, let me see" and would analyze the boards to determine what to do next. She used students work as examples as she asked questions and explained procedures and concepts. Each student had a towel to wipe off their board for the next problem. This technique was also used when Kathy conducted guided instruction and mat activities with the students. It was a way that she could see how students were performing and understanding what she was teaching. Kathy talked about how important it is that she uses the white boards for mathematics teaching in an interview,

> Those boards are really important to me. Because it shows me what they're thinking. I can't listen to every one of them. But I can see what they're doing on their board. I can see how they're thinking and then I know what direction I need to take. First of all I have to figure out why they wrote that. Why did they get that? Where do I go from here? Then I know where they're coming from so how do I solve it? What little piece do they need to have a better understanding of that. That's why it is so important to me to see kids solve problems. I've got to know what they're thinking all the time.

Kathy used the white boards to see what students understood about the mathematics she was teaching. She also talked about what she hears as she is teaching in an interview.

> I look at what I see and what I hear. Sometimes what I hear tells me that they understand but what they show me tells me they really don't and vice versa. I can sometimes see it on their board that it looks like they understand it but if they can't articulate it and articulate it correctly then they don't have an understanding of it. I'm always analyzing what they're doing. I try to get in their heads and see what they're thinking. I know that it helps me know what other examples to give or what to do the next day or how to address the same concept differently.

Kathy asked students questions continuously in all the mathematical activities that she conducted in my observations.

One of the problems of the day asked students to compute a two step problem using subtraction. Kathy read the problem, "There are 19 children at the family picnic. Eight of them are boys. The rest are girls. One of the girls is a baby. How many girls are not babies?" The students started writing on their boards trying to solve the problem. Kathy walked around the floor looking at students work. "I want to know what your brain is thinking," she said. Students were solving the problem in different ways. Some drew circles and crossed them out while others used two ten frames, and others used part-part-whole (PPW) mats. Kathy showed each of these strategies to the whole class. The first one was a PPW mat example. Kathy asked the student if the representation that the student had on her board made sense. "It says that 8 + 1 = 6, is that right?" The student says, "No." Kathy added, "That's why you have to check your work." The student then tried to solve the problem in another way. Another student used two ten frames and put a dot in 16 boxes. He then had 9 boxes crossed out randomly on his board. Kathy responded in the following way. Notice her focus on precision, detail and unique algorithm discoveries of the mathematics.

Kathy: What you did was okay but that's not the way I want to see it. Here's what I want. Everyone look up here. All eyes up here.

Then she draws two ten frames on the board and counts to sixteen as she puts the dots into the ten frames.

Kathy: When you subtract always start with the last one. And you subtract eight. So you go 1, 2, 3, 4, 5, 6, 7, 8.

As she is counting she is crossing out the dots in the ten frames from the last one back toward the first one.

Kathy: You actually are taking them off your ten frame mat or in your head ten frame mat you'll be removing them. So what is your difference when you had 16 –8? What was your difference?

Student: Eight

Kathy: And then what did you do?

Student: I subtracted the baby.

Kathy: So you subtracted the one that was the baby. So now what is your difference?

Student: Seven

Kathy: Okay. So the reason I didn't understand what you were doing is because you did this. And in about a week and a half I will scream at you for this because I'm going to really be picky how you use your ten frame mats. Here's what you did. You subtracted eight like this and then you subtracted one over here and I was having a hard time figuring out what you were doing. If you would have stayed organized and taken off where you left off I could have followed you. Okay.

Kathy continued and said, "Let me see. Oh, that's interesting Tina did this. Tell me if you are allowed to do this." She drew a PPW mat on the board and had three boxes on the bottom instead of the two the students were used to.

Students: No

Kathy: Why not? Can you have three parts?

Students: Yes

Kathy: Have you ever added three numbers together?

Students: Yes (Students start talking and the noise level increased).

Kathy: We don't normally make it like this but Tina did. That's alright with me.

The representation on the board of the student's work showed the three parts of the problem. It had the 8 (for boys), 1 (for baby), and 7 (for girls that were not babies) and the top or whole portion of the representation had 16.

Kathy jumped at the opportunity to show the students multiple ways to solve problems. She talked about the different ways students can solve problems in an interview.

> When I first started teaching and when I was in math classes there was only one way to do the problem. You use the algorithm and that was it. But if you listen to these kids there are ten different ways to solve a problem. Math isn't as cut and dry as I might have thought when I first started teaching.

Kathy encouraged students to solve problems in their own way and analyzed their thinking to determine if that strategy was correct. She explained in an interview,

> They (students) need to see that my brain doesn't think necessarily the same way that someone else's does. But there is another way I can do it that is just as correct. So we need to find that way that works for them. They probably don't get it the way I show it. I want them to always figure out a way that they could be successful in mathematics. It's just a good thing to have lots of ways to get that right answer.

Every lesson that I observed had guided instruction activities, sometimes multiple times in the same day. The problem of the day is one example of Kathy's guided instruction. Most of the guided instruction activities occurred with students having their dry erase boards to explore the problem or mathematics. These activities always occurred with students on the floor in the front of the room by the white board on the wall. Kathy lined students up in two rows of about nine students facing the teacher and the board. Usually a few students would watch the activity from their desk for behavior purposes. Kathy got the students close together when she wanted them to pay attention to something that was important. She would call a "time out" and students stopped what they were doing and went back up to the front to huddle. Kathy gathered the students closely together to get their attention and discussed the directions of the task, expectations, or explored students' reactions to the task. She told me later in an interview,

> My time outs are where we have to get back together and I have to phrase something differently. If I see something wrong or something someone is doing is not correct, I figure that more of them are doing that and we had better talk about it. Get them on the right track. If I don't then they practice the whole math time on wasting their time.

The following guided instruction activity shows how Kathy taught her lessons. Students were asked to put their boards up from the problem of the day and come back up front for a discussion about how to add three numbers. She continuously showed students multiple strategies for solving problems conceptually.

She starts the discussion by asking a question:

Kathy: What do we know about adding three numbers? What do you remember about your brain when adding three numbers?

Student: Add two numbers at a time.

Kathy: Add two numbers at a time. Some of you can do it so fast that it looks like adding all three at the same time but truthfully your adding two numbers at a time. So, we can use our same strategies for adding two numbers we can use for adding three numbers. Who can tell me one of our strategies, one strategy for adding two numbers together? Who knows one?

Student: count on

Kathy: Okay, the count on strategy. Count on 1, count on 2, count on 3. Thank you Lynn you got our brains moving. Makenzie?

Kathy wrote the strategies on the board as the students recalled them.

Makenzie: Doubles

Kathy: Doubles. We're really on a roll now. Okay, Carter?

Carter: Doubles plus one.

Kathy: Doubles plus one. Thank you. I knew you knew this. But when I looked up and nobody had their hand up I was kind of scared. Brady?

Brady: Doubles plus two?

Kathy: We actually don't do doubles plus two. Really what we're doing is doubles and count on two. And really doubles and count on one.

Student: Counters

Kathy: We can use counters. Do you carry counters with you all the time?

Students: No

Kathy: It's not the handiest but you can do it.

Student: number line

Kathy: Number line in your head which you don't have to carry around with you. I like that some of you have them in your head. That way you always have them with you. What's another one?

Student: Ten frame

Kathy: Ten frame. Hopefully it's in your head. Like a dot cube. You can just see it and you know what it is. Give me another one Ethan.

Ethan: count on

Kathy: We have count on one, count on two, count on three. Those are all the count on's. Uhm, Keith.

Keith: part part whole

Kathy: Part-part-whole in your head if that helps you. I don't know but if it helps you we will do it. How about fingers?

Students "NO"

Kathy: Don't have fingers because we have strategies. Yes.

Student: Order property

Kathy: So, if you know that three plus two equals five. Do you know what two plus three equals?

Student: Five

Kathy: Okay, another one?

Student: Pictures

Kathy: Pictures. Draw a picture. We're not going to use all of these today because I'm going to erase them. Easton.

Ethan: Ten, Make Ten.

Kathy: Make ten, good! We are going to use that one today. Most of the games we are playing are make ten. Make ten is one of our strategies. Lynn do you have another one?

Lynn: Use your head

Kathy: Okay, you can have them memorized, use your head, but you actually have to use your head for all of those. Okay, I have a job for you so sit down. Here's your job. I'm going to assign you a partner. I get to decide your partner so you don't have to find a partner. I'm going to give you a math fact adding three numbers together. I need to know the sum of the three numbers but more importantly I need to know what strategies you used. Now I would tell you that when we're adding three numbers you're going to use at least two strategies. You can't just do doubles. Because if its doubles like this (writes 4 + 4 + 3 on board) it probably has something else. It's got to have three numbers. You can use doubles but what else can you use?

Students: (pause)

Kathy: I would use doubles to get 4 + 4. What is 4 + 4?

Students: Eight

Kathy: Then what will I do? I'm not there. I still have a three. What am I going to do with that three? How am I going to get it?

Student: Count on three

Kathy: Count on three. Alright, so if I'm talking about this I would say 4 + 4 + 3 = 11. I used doubles and then I counted on three. Raise your hand if you heard that. Okay, you and your partner are going to have to tell me the sum and you are going to have to tell me the two strategies you used. Now I'm going to eliminate some strategies. That means some of them I don't want you to use. Alright, they are not efficient. Okay, I'm going to take off, we're going to use mental math. So we have to take off picture, order property will work but you will have to tell me which order property you used. Part-part-whole mat. If it works will let you do it. Ten frame will let you use. Number line in your head, we're not going to use number line today. We're not going to use the real number line either. We're not going to use counters. You'll have to figure it out without using the counters. Doubles plus one is that a good mental one?

Students: Yeah

Kathy: We'll use that one. How about doubles?

Students: Yeah

Kathy: Yeah that's a good mental one. How about count on 3? Count on 2, count on 1, Make ten? Yeah these are really efficient mental math things. Mental math strategies. You don't need your fingers to do mental math. You need strategies. When I give you your card you and your partner will share the card and it will have a fact on it. And I'll give you a piece of paper. You and your partner will have to talk about the sum and how you got it. Which one of these did you use and you have to use two of them. You can't add three numbers without using two strategies. But if you think you can and can prove it to me raise your hand and I will come to you.

Kathy picks the pairs to work on the problems together and the pairs go back to their desks.

Kathy: Make sure you talk. Mathematicians talk. You are going to tell me how you are going to solve the problem. What strategies did you use?

Students start talking and working on their specific problem. Kathy calls the students back up front because she sees something that she wants to guide all the students through.

Kathy: Let us pretend that this is what's on my card. On my piece of paper I have to write the sum.

Student: What does sum mean?

Kathy: Okay, the total, the answer, the sum. And then I have to write how did I get the sum. In order to get the sum you have to use two strategies. Now that doesn't mean that Easton is going to give me two strategies and his partners going to give me two strategies. I only need two all together. So, let me tell you how I did this one. The sum is 11 or I could write 4 + 4+ 3 = 11. I know that in my brain that I'm fast, but I can only add two numbers at a time so here is what I did. I used doubles.

Student: Doubles plus one

Kathy: Nope not doubles plus one. Doubles I got four plus four equals eight (wrote 4+4=8 on board). Then I what? What do I do next, Kathy?

Student: eight plus three

Kathy: I did but that's not the strategy I used. What strategy would I use to add eight plus three?

Student: Doubles plus one.

Kathy: Doubles plus one. So, 4 + 4 is eight and one more is nine. That doesn't work. Connor.

Connor: Uhm, three

Kathy: Why three and not one?

Connor: Because there's not a one.

Kathy: Right. I need three more I don't need one more. So, I'm going to use count on three (writes it on the board in words). She then wrote 4 + 4 = 8 + 9, 10, 11. So it's kind of like making dots on the number line. So, I have my sum, I have my strategies, doubles and count on three and I'm done. Put you hand on top of your head if you understand what I want you to do. Do I want you to copy mine? Can you do it?

Students: Yes

Kathy: GO

Kathy was very adamant about expression of mathematical thought in every lesson I observed. She walked around the room listening and looking at students working. A student was adding 6 + 7 on a problem.

Kathy: Okay, you're using doubles plus one?

Student: Yeah

Kathy: What's the double? I'm having a hard time seeing the double. I don't know how to do that. How do you work that?

Student: Because 6 + 6 is 12 and you add one more.

Kathy: Oh, good idea!

She moves on to another group.

Kathy: Did you use a strategy? You have to tell me one of those strategies. Figure out which of those strategies best show me that 5 + 4 = 9. I know that you already know some of these facts but tell me what strategy would work.

These interactions occurred daily in my observations. She was meticulous in demanding that students conceptually articulate mathematical processes and strategies. In various lessons, I saw students sigh, droop, complain, and hesitate to participate having to complete this kind of process for basic facts that were memorized and students knew the answer. Kathy was adamant to have her students conceptualize the mathematical processes in her lessons despite student's dissatisfaction of the tasks.

In another teaching episode, Kathy is teaching students how to use doubles to subtract using fact families. She had the students using Part-Part-Whole (PPW) mats to identify the relationships between the numbers. She was working on a guided instruction activity as a group at the front of the classroom on the floor.

> Kathy: Your Part-Part-Whole has to be a double. It has to be a double. So use the doubles strategy. Because today that is what we're going to be doing. We're going to use doubles for subtraction. You can show me an addition problem. That's fine. Whoa, you want to go there? Ok, that's fine if you can go there. Okay, show me the subtraction one then. Cover it up. Let me see yours Justin. What's your double? What two numbers did you add together that are the same?
>
> Student: one plus one equals two

Here is an episode of how Kathy interacted with the students when they are together at the front of the classroom on the floor. She is conducting a prerequisite activity to help students prepare to complete the worksheet. Kathy had students complete a worksheet every day of my observations except on the day of the test. This worksheet measured the student's ability to execute the skill that was taught in the lesson. This episode is a continuance of the previous episode where students were using the doubles strategy to subtract fact families. Kathy was talking to multiple students when they showed their boards to her in a group activity on the floor at the front of the class. Notice how she was able to externalize and use students thinking and reasoning to guide instruction.

Kathy: Okay, that's right. I didn't see that number right. Okay, one, two, three, let me see. You gave me an addition one, give me a subtraction one. I don't care how big your numbers are you got to be able to use them. Give me a subtraction one. Cover up your board (to another student). You did not show me a subtraction one. You showed me a good part-part-whole. Your part-part-whole is perfect. Your addition fact is perfect. Give me a subtraction fact. Are you doing 10 + 10 (to another student). What is your subtraction fact that goes with that 10. Cover up your board. Lynn let me see yours. Anthony, you need a subtraction fact. Perfect, Perfect, Perfect. Let me see yours again. Okay, clear your boards. Look at your part-part-whole, Casey, when it's a double, these two numbers are the same. Are your two parts the same? Okay, make them the same. I don't care which number you use, make them the same. Okay, I need a volunteer.

Kathy (cont'd): Actually, I need two. We're not going to use this double because I used it and I seen that a lot of you used it. And we're not going to use any number bigger than 10. I need two people to come up and show me doubles on a part-part-whole mat and what a subtraction sentence would look like and an addition sentence would look like. I'm going to chose two so you can talk together in case you need some help. Because two heads are better than one. So Blake and Connie. You should have your caps on so you're not drying out (referring to the dry erase markers students use with their boards). Any doubles but 8 + 8 because I've already done that one. While they're doing that, on your boards, you do one that is not 8 + 8. Show me the part-part-whole. Show me the addition fact. Show me the subtraction fact. The part can't be bigger than 10. I have a lot of rules today. Okay, stop a moment. Stop if you are working. Let's examine these. Let's see if they followed my rules first of all. Do they have a P-P-W mat?

Class: Yes

Kathy: Is it a double?

Class: Yes

Kathy: How do you know?

Student: There's two fives.

Kathy: Okay, there's two fives. Do they have an addition problem?

Most students: Yes

Kathy: Is the addition problem using a double?

Most students: Yes

Kathy: Did they get a subtraction problem that goes with that that part-part-whole mat?

Most students: Yes

Kathy: Is it correct?

Class: Yes

Kathy: Very Good, let's give them a round of applause! Now, show me what you got. Clear your boards and put them up. Okay, I need you back up here. Ten seconds. Let's talk about subtraction fact strategies. Give me one. There should be one real close in your memory. Give me a subtraction strategy.

Student: How about count back 1, count back 2, count back 3

Kathy: Okay (writes it on the board) You are so right. This is subtraction. He is right. I have to count back. I have to go back. There is another one that should be real close to the top of your brain because we've done them.

Matthew: Uhm, doubles minus one

Kathy: Okay. Real close to the top of your brain. What is another subtraction strategy?

Student: Part-part-Whole

Kathy: You could use a part-part-whole mat. We had trouble doing the doubles for the subtraction problem until I asked you to use a P-P-W mat. That PPW mat really helped your brain. That's a good one. Another one?

Another Student: Make ten

Kathy: You can but we got to make sure that you know how to do this one. We'll talk about that one later. That's not a part of today's lesson. But will talk about it. Do you have another one, Javier?

Another student: counters

Kathy: Counters, you might have to put them in your brain because you might not have them with you.

Student: Ten frame

Kathy: Ten frame is a great one. How many of you can picture a ten frame in your head?

Student: I don't need it

Kathy: Some of you don't need it. Okay, what's another one, Mattie?

Mattie: Number line in your head

Kathy: Okay, you can use a number line. You can't use these on a test, can't use our fingers so we got to make sure they're in our head.

Student: Make a picture

Kathy: Usually on a test you can make a picture. Any others that I forgot about? There must be.

Student: Memorize

Kathy: Okay you just memorized it. But when you memorized it you had to use a strategy to do that.

Student: Order property

Kathy: Order property. We'll talk about that one because you're going to have to be able to use that one. It doesn't really work for subtraction though. In addition you can go 6 + 3 = 9 and you can go 3 + 6 = 9. In subtraction, you can do this but you have to flip the parts. You can only flip parts. This one might be kind of hard to do. So we won't put that one down. But if you know which to flip it works.

Kathy: Today we're going to be working with doubles. Just doubles, but you have to remember on a part-part-whole mat these two numbers are the same. You can't have 8 + 8 and 0. Because the two parts have to be the same. So, here is what I need you to do. Finger on your nose. Put your other finger on your chin. Put your other finger on your knee.

Student: I can't

Kathy: Sure you can. And I need you to very quietly, when I tell you but not yet. Okay, take them off.

Students went to their desk and got out their workbook. The students tore out the page that they worked on for that day.

Kathy: I need you to tear out page 583. That is 5 in the hundreds place, eight in the tens place, and three in the ones place. Don't run. Tear it out and put your name on it. And you have your workbook put away. You don't need your paper you just need you.

The students got their workbook page and left it on their desk and came back up front for the instructions.

Kathy: We're only going to do the first page. I've got to see you do the first page. It says addition and subtraction are related. If I write a fact up here is there someone who can write two related facts? If I write a part-part-whole? I need someone up here that can do this. I need all the related facts for that fact family. I want Brady and Shala to come up and help her if she needs it. The rest of you write it up in the air. All the related facts for that family. Okay, move over to the side. Shala, does that look okay to you? What would you do differently?

Shala: (pause)

Kathy: Does she have two addition problems?

Shala: Yes

Kathy: Does she have two subtraction problems?

Shala: Yes

Kathy: Did she only use the numbers 5, 6, and 11?

Shala: Yes

Kathy: Did each fact have all three numbers in it?

Shala: Yes

Kathy: What would you do differently then?

Shala: No response

Kathy: Thank you. Good Job! That was perfect. Let's give Brady a round of applause. Okay, have a seat and I will show you what we're doing. Here we go. (Reading from the worksheet) Addition and subtraction are related we've done related facts. We know related facts. Use addition to help you subtract. Could you use 4 + 4 = 8 to help you subtract?

Class: NO

Kathy: Okay? Let's try it again. Well we better find out. These are all going to be doubles. (Writes on the board) If I know that 4 + 4 = 8 who can come up and write a related subtraction fact? Anthony

Anthony comes to the white board.

Kathy: You're writing an addition fact. Show me a subtraction fact.

Anthony and the class were stuck on what to do. Kathy guided the students through the process again. Then she looked at another problem from the worksheet to model the procedure to students.

Kathy: Here is 7 + 7 = 14. Now we're going to take all of them, remember that mean ol' minus doesn't show up until they're all together. So I want you to look at all of these and you are going to add and then subtract on each one of these. Down here you are going to add and then subtract but no picture. Can you do that?

Class: Um, hmm.

Kathy: Yeah, remember when we were doing fact families and I think it was Lynn who figured this out. He said this is silly they tell you the answer every time. Do you remember that Lynn?

Lynn: (no answer)

Kathy: They would do it like this (takes a fact on the board and uses the whole number to subtract one of the parts). Okay, I need to have you go and do this and when we have checked it we will tell you whether you can go on and do the back page. Doubles as a strategy.

Students then went back to their seats to work on their worksheets. Kathy walked around and interacted with individual students to guide them through the activity. The following sequence depicts one of those interactions.

Kathy: I want to know how you know that. Tell me why 5 + 5 = 10 and why 10 – 5 = 5 are related.

Student: They are the same family

Kathy asked that same question to two other students. She continually interacted with and analyzed student work and thinking through the entire time in every math class. Kathy had a group meeting at the end of mathematics class four out of the nine observations to summarize the lesson. The following episode is a summary of a lesson using two strategies to add three numbers. She brought the students up front in pairs with their cards and asked them to explain the strategies they used to solve their problem. Notice how she uses the student's work to develop understanding of the processes.

Kathy: You and your partner have to come up here and explain your thinking. I want to know how you solved the problem. How you got the sum. Macali, have a seat. Anyone want to go first? Justin and Macali. Show them the card. You show the card and Justin read that.

Justin: I used doubles and I

Kathy: Okay, when you used doubles what did you add together

Justin: 5 + 5

Kathy disciplines a student for not paying attention and asks Justin to repeat what he said.

Justin: I used doubles 5 + 5 = 10.

Kathy: He used doubles 5 + 5 to get 10 and then what did you do?

Justin: I used count on three and got 13

Kathy: Is that reasonable. Is that what you would have done? Would you have added 5 + 5 first and counted on three. Okay, great thank you. Darren and you partner. One of you hold the card and one of you read the strategy. 2 + 8 + 1 what did you do? The sum is

Girl: 11

Kathy: Okay, how did you get that? What did you do first?

Girl: Ten frame (and the girl was talking about what she did)

Kathy: Okay, Stop. She made a ten frame mat and added 2 and 8 and got 10. What she was really doing was this (wrote on board (2 + 8) + 1 = 11) making a ten. She did it by using a ten frame mat. So they took these two numbers 2 + 8 and got 10. Then what did you do?

Girl: We did doubles (student got lost and didn't know how to proceed)

Kathy: There are no doubles. We added these two numbers and got 10. We have one more to add. What is the strategy to add on more?

Girl: Double

Kathy: Okay, let me show you what you have already done and maybe I can help you. You added the two and the eight and you got ten. Right? Then you have 10 + what?

Girl: One

Kathy: Okay, 10 + 1. Which one of those strategies would you use for 10 + 1? Darryl

Darryl: Count on one

Kathy: That's it count on one. You should have said, I added 10 + 1 I counted on one. So what was your sum?

Girl: 11

Kathy: Okay, so when you got 2 + 8 = 10 you have no doubles there. A double is like 4 and 4, or 6 + 6. You don't have a doubles here to add. Does that make sense?

Kathy worked with the pair's cards and how the students used the strategies to add three numbers. On other summaries Kathy called the students up real close together and asked questions like "How can knowing doubles in addition help you do subtraction? What's in a name? What's a name for 4? What's another name? How does a part-part-whole mat help you? Students would then articulate their understanding of the concepts and procedures of the lesson.

Key issues.

Mathematical expression. Kathy was adamant that students could effectively articulate the mathematics that they were doing. She verbally asked students to tell her what they did or what strategy they used to get the answer continuously. She felt that her students did not understand the mathematics she was teaching if they could not correctly articulate verbally back to her what they had done. In the two step problem, Kathy analyzed three ways students solved the problem and showcased them for the class. She asked questions to model for the class the process of each student's thinking asking questions to clarify and develop concepts and procedures. Understanding of the mathematical strategies and processes was paramount in every lesson. The focus in Kathy's lessons was always on understanding of processes and concepts used to solve mathematical problems and to be able to communicate those understandings verbally to the teacher and the class.

In every lesson, except for the timed fact tests, Kathy expected the students to use and show strategies to solve the problem even though they knew the fact from memorization. Kathy was careful to use the appropriate mathematical terminology in her teaching of mathematics and expected students to use those words. Kathy corrected or developed student vocabulary continuously throughout her teaching. She defined mathematical terms for students, instead of doubles plus 3 she corrected the student to say doubles and count on three, and when a student said count back for subtraction she rewarded the student with positive affirmation and used his terminology to reiterate the subtraction process. She often would restate her question or statement to distinctly use the proper vocabulary or process to make clear what was happening. Kathy corrected a student when she said she used doubles to add 2+8+1 = 11. “What she was really doing is making a ten (2+8=10).”

Directions to assignments, changing tasks, and procedures were articulated clearly multiple times to help guide her students to achieve deep mathematical thought in her first grade classroom. Kathy would ask a series of questions of students’ work to teach the process and model the steps they needed to accomplish the task. For example, she asked students the strategies they used to find the sum of three numbers. She first asked what the sum was, next she asked for the first strategy, and then the second strategy emphasizing the mental process of adding using two strategies.

Immediate feedback. Kathy used white boards in every lesson that I observed. Each student would have a board, towel, and a dry erase marker to work problems. She watched students work their problem out on the boards to inform her of how they were choosing to solve the problem. Kathy used whiteboards to externalize what each student was thinking and analyzed the level of mastery of each student's understanding of the process or strategy she was teaching. This pedagogical decision provided a snapshot to Kathy about the level of understanding that each students had on multiple levels of mathematics in a timely manner. In the doubles to subtract using fact families episode Kathy checked each students work by seeing how they worked the problem. She noticed that some students were not using a doubles strategy and was able to use the work students developed to teach the class specific skills that were lacking. Students were giving addition facts when the problem called for subtraction facts in another episode. Kathy was able to respond to her students to develop needed concepts and procedures. Kathy mentioned to me frequently that it was important to her that she knows what the students are thinking. She mentioned to students in multiple lessons that she had to know what they are thinking in their "head" all the time. This technique allowed Kathy to determine what was understood and what needed attention for the students to be successful in the lesson.

Active interaction. Games were used in every lesson that I observed. Most of the games were base ten games although other games were played that developed other mathematical skills. Kathy would bring out another game for the class if time permitted. She wanted students to actively interact with the mathematics that they were learning. She had hundred's of games and manipulatives that she had collected over the years that completely filled a coat closet. The games provided another social aspect to mathematics. The students had to count usually two numbers and work with one or two others students to use the counting to play a game.

A lot of talking back and forth among students occurred, mostly about using the math to play the game. This process allowed students an opportunity to explore mathematics outside the traditional educational setting. Kathy added or developed new games and created songs that were related to mathematical concepts to make the subject fun, interesting, and captured students' attention longer. Games were also used when students completed a specific task and were waiting for the others to finish. Students enjoyed the games portion of math class and never complained about having to add numbers. Kathy researched which games helped students learn specific skills and told me that each game must have a mathematical purpose in her classroom.

Student processing. Kathy had a clear idea of what she wanted to teach for each lesson. How she taught the lesson was determined by what she saw and heard from her students as they interacted with the mathematics. She had her students work on mats, boards, and used manipulatives to see what students were doing and what they were thinking. This told her about what the students knew and did not know about a process or concept they were using. Kathy also asked questions continually throughout all of the sections of the mathematics block. I counted how many questions she asked in one random lesson and counted over 100 questions. I would consider that lesson to be a routine set of questions that she asked in one lesson.

Kathy was meticulous about how students were processing and using what she was teaching in the lesson. She asked multiple questions to the same student until she was confident that she understood what the student knew about the concept or procedure of the problem. Kathy compared what she was seeing to what she was hearing to analyze deeper what the student's understanding was. The information that she observed from seeing and hearing students independently work math problems determined how she proceeded with the lesson.

Kathy would call a time out and have all the students stop what they were doing and come up front when she felt that there was confusion about what to do or how to do it. She used time outs to reteach a skill or guide the students in how she wanted them to accomplish the task. Time outs were called to reteach concepts and procedures. They were also called to reiterate the process that she wanted students to use and made sure they stayed on the right track. On occasion students wanted to just put down the answer and move on. Kathy would call a time out and was adamant about showing the process of how the student got the answer.

Multiple ways. Kathy looked for and showcased how students solved problems differently. When a student would solve a problem in a unique or creative way she would make a point to show the class what that student did to solve the problem. She usually asked questions to see how other students are processing the different procedure. Kathy wanted students to know many ways to solve the same problem. She frequently would ask, "What's another way to solve it? Give me another way?" Showing students multiple ways to solve problems and using student examples when possible was intentional in Kathy's teaching. Some of the multiple ways to work problems were not planned but explored like the three part PPW mat discovery. She wanted students to know multiple ways to solve a problem and allowed each child to use the one that was most comfortable. Kathy told me that students need to know that there are many correct ways to solve a problem and not everyone thinks the same way. "If you listen to these kids there are ten different ways to solve a problem," she said. Kathy would analyze a student's process to see if it was mathematically correct when a student showed a unique or different procedure to solve a problem. She asked herself if this procedure works in all cases and if the procedure was an effective or efficient strategy and then would showcase the procedure or concept to teach the class creative ways to think about mathematics and solving problems. For example, Kathy explored a student's use of three parts in a PPW mat. The students did not think it was right because they had not seen it before. Kathy used a series of questions to show the student's concept and altered use of the tool was valid.

Participant: Callie Hendricks

Context. The trip to Callie's classroom took about 45 minutes by way of the only Interstate highway in the state. The Interstate had three lanes going in each direction. Callie's classroom was in the largest city in the state. There was mainly farmland and no other cities directly on the Interstate between the cities. It was early spring and the landscape and trees were filled with fresh color. The view was overshadowed at times with road construction. Crews were working on the roads close to the cities at the beginning of the trip and when entering the metropolitan area where Callie taught. There was light traffic between the two cities and I never experienced a traffic jam and generally traveled the speed limit of 75 miles per hour.

There were multiple industrial sites on both sides of the road as I came up to my exit off the Interstate. Two large national chain motels were adjacent the exit I took. The next couple of miles were mostly industrial buildings filled with commercial businesses. I then came to a newer residential subdivision on the left with a newer shopping mall on the right. As I came up to the light and turned into the residential area I noticed that the houses were well-kept with professional landscaping and manicured. The yards were rather small and the houses were packed close together. One section of the subdivision was "cookie cutter" designed where all the houses were similar to each other. The rest of the large subdivision had homes with different architectural designs. There were some ranch homes but mostly two story Victorian style houses. There were a lot of cul-de-sacs throughout the subdivision. As I entered the subdivision children and school signs were found on every block around the school and entering the subdivision. The first sign I saw read, “keep kids alive, drive 25” and had a picture of two children playing with a soccer ball. These signs were securely attached to the metal poles on both sides of the street used for lighting.

The school was five years old and nestled into this suburban neighborhood. The all brick structure was rectangular and had homes built around all the sides of the school. The grounds for the school were rather small with ample room in front of the building for parking that included a semi-circular driveway to let children off and for buses to park. On the front side of the driveway was enough pavement to park numerous cars. As I entered the driveway there was a turn to the right that went to the side of the school for a couple rows of parking spaces for staff to park. There was a small playground behind the side parking lot although I never saw this area being used when I conducted my observations. Students used the narrow rectangular playground in the back of the building. Residential homes were directly behind the school and were fenced off. There were a few sidewalks that allowed students to access the school from the back between houses and through the fences. There were four doors in the back of the building for classroom access to the playground. State of the art playground equipment with large slides and climbing apparatus were housed on both playground areas.

The main entrance was in the front of the building on the east side. I walked down a double wide sidewalk by a couple of metal benches that led up to the building from both parking lots. There was one flag pole that prominently displayed the American flag.

The building had well manicured shrubs that lined the front of the building. In the center of the building in front was a space that housed an outdoor garden. Two portable buildings for additional classroom space were located on the west side of the rectangular building.

As I entered the building I noticed that the school was as clean and organized as the subdivision it was nestled into. The floors seemed to glow and everything including the windows was spotless. The office was directly in front of me with the principal's office directly behind the counter where the secretary usually was. I checked in at the office and had to enter my name into a computer and put on a visitor badge each day I came. There was no security guard present and I usually didn't talk with anyone after the first couple of days except to wave at the principal if she was in her office. The halls were wide and classrooms were on both sides. As I walked toward Callie's room I noticed a large library with state of the art computer equipment and center. There were special keys for entry, motion lights and sensors located throughout the school. Character pillars were sequentially placed and student work neatly organized in the orderly hallways. Modern amenities were everywhere.

As I entered Callie's room, I saw four large groups of desks in a square shape in the middle of a large classroom. There were six desks put together to make a square and four groups of desks that housed up to six students in each group. As I walked through the front door I noticed another door across the room that led into the adjacent classroom. Storage cabinets and shelves were everywhere and filled with materials. On the wall to the right were a huge whiteboard and a bulletin board filled with math terms. On the opposite wall was another math bulletin board with the calendar, clocks, place value holders, and a hundreds chart. This board was not used during the times that I observed. A new television set was hanging from the wall in the front and there was an overhead projector that had math pockets completely around the machine. The teacher's desk was in the back by the door to the other room. There was a large rocking chair by the desk. In the middle of the back of the room was a large rug where students went for group instruction. In the opposite corner was a large semi-circular table where the teacher could work with a group of students. That is where I sat to conduct my observations in the classroom. On the other wall facing the hallway were hangers and cubbies for the student's coats, bags, and books. There were small windows by both doors in the room. The room was large so students could easily move around the room and their desks.

Teacher. Callie loved to play school when she was a child. She told me about her childhood play in an interview.

> I always played school when I was little because I loved the paperwork part of it. I loved having no books and writing down great things. So, instead of Barbie's that little girls usually play with, I would have sticky notes, rulers, clipboards, and pens. As a teacher I could have all that plus work with kids, and so it worked out for our family.

Callie wanted to be a teacher but struggled in math and did not like it when she was in school. She talked about her experiences in mathematics during an interview.

> It was hard and I didn't like it. In high school, same thing. In college, I had to take the math exam three times to be able to get into math and remembering at night just sitting there trying to do work and studying it and none of it made any sense to me. Adding and fractions and all that algebra stuff. Not a clue. So it was an awful experience. I did not like it at all.

Callie has taught for eight years at the same school district, seven as a third grade teacher. She taught second grade for one year. She is a member of the local education association, NEA, and NSEA. Callie developed confidence and skill in mathematics through professional development opportunities in her school district and taking additional coursework at the local universities. She has a bachelor's degree in elementary education and has achieved a master's degree in a Cadre program that focused on elementary teaching. She has also achieved 36 additional credit hours above the master's degree. "I take classes that interest me; special education, language arts, and math," she said. I asked her about her mathematics training in her master's program during an interview.

> Math was my worst subject going to school. It doesn't make sense to me especially the really abstract higher level math. I still don't get it. So, I never liked math at all. Math in college was awful. Because the way I was taught in school was memorizing and just I'm going to tell you what it is and you memorize it. And I never understood it. And those are not my beliefs in teaching.

Then she talked about a breakthrough for her in mathematics when she volunteered to be her school's mathematics toolbox representative for the district by chance.

> I don't think I started liking math until I was on the math toolbox which is our district's math curriculum. We as a group have helped write the assessments, done a lot of research for best practices in math and that's when I finally started to understand it. So I think I have a better understanding on how to teach kids math. And I think kids love it. They have fun in math. We don't even keep books in their desks. We don't pull them out and do 1 through 50 today and move on. It's a lot of hands on and discovering and talking and discussing and having fun with it. Toolbox was the turning point.

Callie also took a mathematics class in college that taught her about how she could teach mathematics differently.

> I took a class a few summers ago that was a lot of hands-on, that was a lot of math games, and I guess that I didn't realize that you can play tons of games and the kids think that they are playing games and don't realize that they're actually doing math.

> I could have fun with it and you can play games and still learn and it doesn't have to be sitting in your seat memorizing things.

Some of her hobbies are to read educational books and be a part of developing mathematics through participation on committees. Her favorite author is Marilyn Burns where Callie commented, "That talks about how kids need to play. They need to discover and choose manipulatives. They need to be involved and understand it versus telling them something. Discovering it makes them accountable. Makes them learn more."

Callie is a third grade representative for her school and teaches her peers on mathematical developments, research findings, and new programs. She has served on the MODEL assessment committee that researches best practices to accommodate all students in the classroom. Her toolbox experiences include: third grade representative where she presents findings of toolbox to teachers at her school during inservice meetings each month. She helped select the district's new math textbook series. She piloted the series last year and presented findings to the principals and administrators at the central office.

Callie helped write the UBB guides and assessments for mathematics in the third grade. The UBB guide is a detailed document of what mathematics should be taught and focused on, the timeline for specific curriculum, and deciding what should be mastered, proficient, or introduced.

Callie's mathematical journey included the inspiration of another teacher who is now the leader of the toolbox. She met this person as a teacher in her school and learned from her about teaching mathematics. Her mentor showed her how to teach students for understanding and shared her research findings about best practices. Now Callie has another teacher that is learning from her.

Survey. Callie Hendricks took the CKT-M (content knowledge for teaching mathematics) instrument from the University of Michigan's learning mathematics for teaching project. The following table shows how Callie scored.

Table 4

CKT-M Scores for Callie Hendricks

Construct	IRT Score
Numbers & Operations	0.0682
Patterns, Functions, & Algebra	0.2480
Knowledge of Students & Content	0.8840

She showed her work to the problems on the assessment and I noticed that she had the right logic to three problems that were correct but selected the wrong response. It seemed to me that Callie may have missed some easy responses because of the anxiety that she experiences from taking mathematics assessments. One question asked for a point on a number line that was closest to 7/16 x ½. Her answer was ¼ but selected point B which was a little below half. She had the proper logic and was able to determine the answer correctly but selected the wrong response. She also missed two responses to students' multiple answer solutions to problems that she would have allowed in her teaching in the classroom. Callie was adamant about allowing and showing students multiple ways to solve problems in my observations of her teaching.

Callie's strength in the survey was in the construct of knowledge of students and content. She had an IRT score of .883. This indicated that she has exceptional skill in assessing students' mathematical understandings and is able to determine where a student needs additional development. She scored above the mean in all three constructs.

Callie's teaching. In this section I have integrated observations and interviews in a description of Callie Hendrick's teaching. Also, within the writing of her teaching I used examples drawn from multiple lessons to show a more complete picture of her teaching. I observed twelve consecutive lessons that usually started around 10:30 a.m. each day. I observed the main mathematics class that was scheduled for one hour but frequently went longer. Callie told me about how she also taught math at other times throughout the day during an interview.

> It's interesting because in math you have your math block and then in the morning we do math and at the end of the day we do calendar math. It's like we're doing math like four or five times during the day.

I only observed her teaching the math block portion of her mathematics teaching. I would frequently notice that a lesson or concept had been developed or worked on when I arrived the next day to observe. I conducted three semi-structured interviews during the twelve days I observed Callie in her classroom. Callie taught mathematics exclusively through problem solving and a problem would usually take at least one whole class period to explore. She also taught in series of problems with related concepts and procedures. I chose segments of Callie's teaching episodes that depicted holistically how she taught mathematics, what she focused on with her students, and how she interacted with her students to develop understanding. I used this strategy of presenting her case because she taught her lessons holistically without breaking them into easily identified series of activities.

I came a few minutes early each day and set up in the back of the room at the semi-circular table. The class would be finishing a lesson in language arts as I entered. Students would come to the front of the classroom and sit on the rug to start mathematics. I purposefully did not interact with the teacher or students throughout each lesson because I wanted the experiences to be authentic as if I were not there.

The unit that Callie was teaching during my observations was on fractions. Most of the lessons were introducing fraction concepts and building number sense. I observed lessons on equivalent fractions, comparing fractions, adding fractions, and fractions of a specific number. A manipulative of some sort was used in every lesson. Some of the manipulatives were: Hershey bars, brownies, geoboards, pattern blocks, paper cookies and brownies cut in fractional parts, tiles, and other games. Callie did not use a textbook with students in any lesson that I observed. She explained:

> Basals are easy to teach from. They're easy for teachers to plan and teach from. Textbooks focus on teachers. But when you have no basal then your teaching focus is on the kids and their needs. We've probably pulled those textbooks down about four or five times the whole year. Maybe a lot of times I'll pull those down if I need five or six problems to check for understanding so I can check and see how they were doing. I use "Investigations" and that is what it is. Kids are investigating and discovering. It's the hands-on discussion and interacting method.

Callie taught with a co-teacher. During my observations the co-teacher was more of a helper than a co-teacher. She would read a book or provide support when students were working independently. Callie talked about their co-teaching of mathematics in an interview.

> When I'm co-teaching I start to just take over. I have to apologize to her all of the time for that. Now in reading we break into groups so it's not an issue. This is the first year we co-taught math together. Last year we did reading and writing and that was no problem. But math I don't know if it's because I feel a connection to math like it's mine now. With toolbox and piloting the textbook and presenting it to administration I just have that connection and it's hard to give it up.

I selected five teaching episodes of Callie's teaching in her classroom to show how she used activities to engage students in the exploration of mathematics. One episode shows how she used multiple activities in a lesson to guide students in multiple skill sets through modeling and dialogue. Another episode showed dialogue between two students in a group and their interactions with Callie. I wrote the episodes in long detailed dialogue because it is important to witness how she interacts mathematically with the students.

Callie started every lesson with an activity to engage students in mathematics throughout my observations. She provided manipulatives for each student in every lesson to use in solving a problem or question she developed allowing students to interact with and discover mathematics. Callie had students work on these activities either individually or in small groups of two or three. She followed these activities with discussions about the mathematics the students were learning. Callie frequently would use the statements "tell me more" and "turn and talk" with the class to actively involve the students in the process of understanding the mathematics. She told me why she asked students to turn and talk in an interview.

> When they turn and talk they all have great ideas but they don't always want to share them. So by having them turn and talk it gives everybody a chance to talk and hear one another versus just a few who raise their hands. By letting them talk in groups or to a partner helps them also with asking and answering questions because they know it's not just them but they'll have that chance to have that interaction and dialogue and clarify anything before answering.

The first lesson was an introduction to fractions. Callie read the book "Hershey Fractions" to the students. The book talked about what a fraction is. She gave each student a Hershey bar as a manipulative and reread the book having students use their bar to simulate what the book was representing. Callie noticed in the middle of the book that students were confused and unsure about all the concepts that the book was explaining about fractions. She stopped the lesson and talked with the students about their feelings. Callie told the students that it was alright to be confused and that it will make more sense as they work through the whole unit. She compared it to other math topics that the class had struggled with throughout the year. The students were receptive to her explanation and it relieved their feeling of inadequacy. Callie gave distinct instructions to students about what they were supposed to do and how to use the manipulatives in every lesson. Callie used a calm and playful voice in her teaching that provided a safe and secure environment for students to explore and discover mathematics. The following episode shows verbatim interactions of the Hershey lesson with her students. Notice how precise and unambiguous she was in providing tasks for her students.

Callie: You will put your candy bar on top of the wrapper like this. Open it. Make sure it is facing up so you can see all the little pieces. That's perfect. See Mike. He has it in front of him and he is not touching it. Is everyone ready? If you think you are ready and have your candy bar in front of you I am going to start reading. Eyes up here. And we're going to read a page and we will work with our candy bar to show that fraction. Does that make sense? So I would like your eyes up here and I'm going to read a page and we will talk about it if we need to and then you will be able to show it with your candy bar. So first you have one candy bar.

Callie started to read the book.

Callie: Milk Chocolate. Uhmmm. First of all do you know where milk chocolate is made? Where Hershey's is made? This was a question that you remember that was made over the intercom. The first teacher that could run down and tell them in the office what is was.

Les: Hershey, Pennsylvania

Callie: Exactly, Hershey, Pennsylvania. And they had us learn about Hershey, Pennsylvania and my daughter Kelsey comes home and says, Mom can we go to Hershey, Pennsylvania? For what? To go get some Hershey candy. No, but we can go to Target and get some. Okay, you have yours unwrapped so you can see that if you have not. If it wasn't broken apart. When I opened mine it was broken but it looks like that most of yours are together. So when you look at your candy bar. I'm glad that I have all eyes up here. Remember that I'm reading first and

Callie continues: then you are going to be able to break your candy bar apart. So you have one whole candy bar. One whole. We can show that by just the number one. One whole. Now Miss Johnson talked about other ways that you can show one whole. Do you remember what one is?

There was no response to the question.

Callie: Let's see if you can turn and talk with your partner and see if you can figure out another way to show one with your partner.

The class started talking together about the question and the noise level went up. It was clear that the students enjoyed talking to each other when the teacher asked them to turn and talk.

Callie: Okay, did anyone figure it out? Another way using fractions that you can show one whole.

Boy: Twelve twelfths

Callie: Twelve twelfths. Why does that represent one whole?

Boy: Because I have 12 pieces and the candy bar is 12 pieces.

Callie: Exactly, so he could write it as 12/12 and that would be the same thing as one whole. Exactly, you're right. Okay, now it says if you break them apart. Looking up here. If you break them apart how many little pieces am I going to have? I'm going to have?

Class: Twelve

Callie: Twelve little pieces. Go ahead and break all your pieces apart.

Calli continued the lesson using this distinct method of detailed instructions, discussions, and student interaction and exploration in every lesson I observed. She selected manipulatives that grabbed the students' attention and gave explicit directions on how to use them. The next lesson used brownies and students had to develop strategies to make their pieces equal and prove it. The same concepts were used in the third lesson using geoboards. The sequence of lessons continued in order with: creating and comparing fractions, discovering many ways to add fractions to make a whole, finding the fraction of a number, using division to find fractional parts, using pattern blocks to determine what fraction of the whole is each piece, substituting fractional parts into wholes, equivalent fractions, and using geometrical pattern blocks to design shapes and to be able to explain how many wholes and fractional parts were used.

Another example that Callie used to engage students actively in the mathematics she was teaching was when she had students make designs with pattern blocks with the goal of having ½ of the design being yellow. The task was to build whatever shape the student wanted with the blocks as long as half of the design is yellow pieces (wholes). Other colored pieces represented 1/2, 1/3, and 1/6. The students had to add the pieces and compare them with the yellow pieces. An initial activity preceded this activity where Callie had students compare the pieces to figure out what fractional part each piece was to the yellow piece. She did not tell the students what fractional part each color was. She used an overhead projector and see through color pattern blocks to help students discover the value of the different colored pieces. Then she had another activity where she modeled to students how to determine if ½ of a design is yellow. Callie started the activity by displaying a design and modeling for the students the mathematical task of the activity. She presented a design with 2 yellows (whole), 2 reds (1/2) and 3 blues (1/3) triangular pieces and said:

> Callie: What fraction of this design is yellow? I'm looking at the whole design. What fraction of this is yellow? (No response) Turn and talk and why?
>
> The class noise increased as the students were trying to share what they thought with each other for about two minutes.
>
> Callie: Okay, back to me. Is this also one half? If it isn't what do you think it is? Ben thanks for being confident. (she calls on a couple of students to get their conjectures).
>
> Ben: 1/3
>
> Austin: 1/4

Sam: 2/4

Callie: Can you give me a reason? Sam you had 2/4ths.

Sam: There is four wholes all together and there is two that are yellow.

Callie: What do you mean you have four wholes all together?

Sam: You put the two reds together to get a whole. You put the three blue ones together to get a whole and there's two whole yellow ones.

Callie: So you're saying if I go like this (she manipulates the pieces on the overhead) like that? So how many of you noticed that when you first started? No response. Could you all get out two fourths and one half and could you please compare them for me?

Students got the pieces out and started to compare them. It is quiet for about a minute and then a couple of students say "Oh Yeah!"

Callie: Wait. Wait. What Tim?

Tim: 2/4ths put together equal ½

Callie: 2/4 = 1/2 (wrote it on the board). So, if I have these you are saying that 2/4 is the same as ½. Let's look at this as four separate ones and if I'm talking about 2/4ths than I'm talking about this much. And that's also the same as a half. So was this design also ½ yellow? (she demonstrated as she talked by manipulating pieces on the overhead)

Class: Yes (unconfidently)

Callie: Your job today with your partner. You are going to get typing paper. Just one. You will fill up one and then you can get another. But you work together with your partner. You're going to get pattern blocks. The same ones that you've been using the 1/3, the ½ and the whole. You are going to use pattern blocks today and you are going to create what we call a ½ yellow design. And we call them ½ yellow designs because the design that you make half of it has to be yellow. Half of it has to be yellow. So could I make a design that looks like this?

Class: Yes. No. Yes. (half of the class is split and changing their minds)

Callie: If I make a design like this it has to be ½ yellow. Have I created half yellow? Is half of it yellow?

Class: No

Callie: Is half of it yellow? "NO" Then does this design work? "NO" What if I did this. Would this design work?

Class: No (after some thinking and a little more confidence) with a couple Yes.

Callie: Turn and talk. Why or why not? Back to me. Ben what did you find out?

Ben: Then there would be one more red one because a blue one and a green is a half.

Callie: And I like the way that you're talking today in math terms like the way that you are deciding to move your pieces. I had to move my pieces to understand what you were saying. Nice Job! Ben's saying it doesn't work. He says we still need another ½. Will this design work? (she puts another design on the overhead)

Class: Yes (more confidently)

Callie: Why? Justin, why does that work?

Justin added the pieces as Callie showed proof of how the pieces equaled ½ on the overhead. She then went on to give distinct instructions to the students on how to complete the activity. Callie gave students an additional activity in multiple lessons to give them an opportunity to show their new level of understanding of the concept after a modeling experience and discussion.

This next teaching episode shows how Callie wanted students to interact with and discover mathematical number sense using fractions. This lesson used fractional pieces of paper to develop understandings of how adding fractional parts make a whole. She started the lesson by teaching students how to measure equal parts. She gave two strategies to determine how to measure and cut out a fractional piece of a sheet of paper: using a ruler and folding. She used paper to model both strategies using 1/6th parts. The ruler produced six long strips. She then folded the other paper in half the long way and folded that three times. This produced six shorter pieces that were twice as wide. She asked the students if both represented 1/6th. Callie asked them to prove whether it was the same or different because students were unsure about the answer because the shapes looked different. Austin, a student, conjectured that the bottom of the long strip was two inches and the bottom of the wider piece was four inches. Callie developed the proof by cutting the long in half and putting it beside the other ½. Then she put it on top of the wider piece and they were the same. She told the students that because we started with the same amount as the whole we can have different shapes of 1/6th and it will be equal.

The next part of the lesson was identifying what fractional part a piece was, cutting one piece off the whole and labeling both pieces with the proper fractional part. Callie presented a piece of paper that had a ¼ of a piece cut off. Her discussion continued as she explored labeling of the two pieces and asked the students for mathematical reasoning to their answers and modeled strategies to identify the fractional parts. The students explored other papers (wholes) that had different fractional parts: 1/3rd, 1/4th, 1/6th, and 1/8th and labeled them. Then she asked the students which was the smallest and which was the largest fraction.

The lesson continued by using different fractional parts to make a whole. Students were given fractional pieces of a sheet of paper that were labeled with their fractional part. Students were to place these parts on a sheet of paper to completely cover it up. One student had a half piece and two 1/4th pieces to cover her paper. This produced the fraction sentence of ½ + ¼ + ¼ = 1. Callie noted the equivalent fraction 1/2 = 2/4 to the class. Students then used other pieces to discover other ways to use fractional parts to make a whole. They used pieces of ½, ¼, 1/3, 1/6, and 1/8 to make a whole.

She finished the lesson by having the students come back up to the rug and share their discoveries with the class. Callie wrote their responses on a paper chart labeled fractional facts as the students presented their discoveries. One half of the sheet was labeled halves, fourths, and eighths and the other half was labeled halves, thirds, and sixths. Callie discussed the responses whether they were correct or incorrect with the students. She wanted to make connections between the fractions and build number sense. For instance, a student responded 1/6 +1/6 + 1/8 + 1/8 + 1/8 + 1/6 + 1/8 = 1. Callie added that 3/6 = ½ and 4/8 = ½. So we could write another fraction fact 3/6 + 4/8 = 1. After these experiences, students got real excited about manipulating fractions to equal one. Students continued to discover ways that they could add fractions during the next week on their own time and at home to find more ways to add fractions to the fraction fact wall.

Students had experience with and interest in the problems that Callie created to help students focus on and understand the objective. She allowed students to teach each other through expression of their thoughts, discoveries and ideas during discussions. One lesson had the objective to find the quotient of a fraction of a number. Callie first used an example of ½ of the class. Notice how she used student's thinking and ideas to guide instruction.

Callie: Today you are going to figure out the fraction of a number. What this means is you are going to get a shape usually a square or rectangle and then you're going to divide that into fractions. That is just one thing. You just used one piece of paper, one square, just one geoboard, just one cookie. We took one cookie for Braden's birthday and divided it. For example, let's think about this. We have 22 students and we have 2 gone today. So, we have 20 students. So we have 20 students today. If I wanted to divide you in half could I do that?

Class: Yeah

Callie: Yeah. How many groups would that be?

Girl: 2

Callie: It would be two groups. Raise your hand if you could tell me how many students would be in each group? If I want to find one half of my class there is twenty students. How many students would that be, Sam?

Sam: 11 (most students raise their hand)

Callie: Eleven. How do you know?

Sam: No, because 11 and half is 22

Callie: Okay. Watch what I'm going to write (teacher writes ½ of 20 = on the chart paper) because I want to know what is half of twenty. So 20 is my total number. Because I have 20 kids in here today, correct. So one half of 20 is how much?

Class: 10

Callie: Ten. We are going to take a group of something like our students in our class. There is more than one of you and put you into one group. And we count that as one whole. So you are one whole class today. But I can still divide you in half. And I can even divide you into fourths. I can divide you into fifths. The way that we are sitting in our classroom too we are divided into groups. If you are sitting in equal groups that would be equal. Just like when we divided our square or cookie my groups have to be equal. Just like when I divide you I am looking for equal groups. We are going to work with some objects today and they are eggs or candy. You are not going to eat the eggs or candy while we are working with them. You can eat them at the end. What we're going to do is take a group of them and we are going to figure out what's a half of that, what's a fourth of that. Just like we took a whole class of 20 and divided them in half and found out that there was 10 in each group. So we're going to practice a little bit together and then you are going to work with a partner to answer some questions and then you are going to work by yourselves to answer a couple of questions.

The teacher told students who their partner was and had them go back to their desk with a set of eggs to work with. After her initial skill activity she modeled using eggs as a manipulative to solve problems of fractions of a number.

Callie: Each of you are going to get a bag of Easter eggs that you are going to use to do similar problems. So would you make sure that you're full attention is up here. First of all we are starting with 18 gummi bears. So kind of like a class, think of it like a class. One full group it just that there are eighteen that make up that one whole group. I bet that you can tell me this. (She writes ½ of 18 = on the overhead) One half of eighteen. If thinking of my full group of 18 gummi bears. I want to divide them in half. If I'm dividing them into half, how many groups do I have?

Girl: 2 groups

Callie: 2 groups. Do you agree with her?

Class: Yeah

Callie: I'm just going to write that down here. So I'm dividing in half so there's two groups. How many bears are in each group, Lexie?

Lexie: 9

Callie: Nine. How do you know?

Lexie: Nine plus nine equals eighteen

Callie: Nine plus nine equals eighteen. Actually I laid them out in a way that represents what I just did, correct. I have one, two, three, four, five, six, seven, eight, nine and then nine. You just took 9 + 9 = 18. Did anybody use a different strategy? Then using nine plus nine. First of all did anyone use my gummi bears picture because it was already laid out. Jeremy you used it. Branden, what did you do?

Branden: Nine time two

Callie: How did you know nine?

Branden: Because there is nine in the top row and nine in the bottom row.

Callie: So you used the picture also. Okay. So you used like an array. Two groups of nine. Two groups of nine. Two groups times nine. Good strategy. Taylor, what did you do?

Taylor: Counted by twos

Callie: So you counted by twos. So tell me how did you know when to stop.

Taylor: Counted the row.

Callie: Oh, you counted the row. You just used the strategy of counting by 2. What if I wanted to find out (as she writes it on the overhead) 1/3 of 18 = ____ ? So 1/3rd of 18. So when we looked at fractions 1/3rd of something it was one out of three. I remember the bottom number represents my whole, right. That bottom number. So if I have three and I take my 18 and put in groups of three. I'm dividing 18 into groups of three because 1/3rd tells me that. Can everybody see that 1/3rd where I'm getting that 3? What does my top number mean, Claire?

Claire: One group

Callie: One group. So if I'm going to take my eighteen and divide them up into groups of three first that top number represents one group. How can I do that, Sam?

Sam: Well you can just divide the bottom number by the top number?

Callie: Okay, so taking three gummi bears, is that what you are saying? So there is my three groups is that what you mean?

Sam: Yeah

Callie: So then what should I do because I still have all of these left?

Sam: Then you keep adding to them to the groups.

Callie: Tell me then if I'm doing this right. Adding one at a time (she manipulates the gummi bears on the overhead). So I have my three groups, correct. So I need to find out 1/3rd of 18. So if I think back to what Claire said, she said that when looking at that 1/3rd or one out of three then that would mean one group. What would 1/3rd of 18 be?

Girl: Six

Callie: Six. Where did you get the six?

Girl: One group

Callie: So you counted up how many gummi bears were in one group and that gave you six. So 1/3rd of 18 = 6 and I double check my counting to make sure there was six in each group. So that way they are equal. And they are. There is six in each group. So did anyone do this a different way?

Boy: I went 6, 12, 18

Callie: So you counted by six, twelve, eighteen and you knew that meant that each group had six in them. If I wanted to find 1/6th of eighteen how many groups would I be making? 1/6th, Faye.

Faye: (unsure) six

Callie: Six. What if I wanted to find out 1/8th of 18? How many groups would I make, Laurel?

Laurel: Eight

The teacher moved to an activity where students worked in pairs to see if they could use what was taught. The teacher handed out bags of eggs and had the students count the eggs to make sure they had eighteen.

Callie: The first thing I want you to do is to find for me ½ of 18. Will you show me on your papers ½ of 18?

Callie walked around the room looking at what students were doing with their eggs. She asked a student a question.

Callie: One half. If I want to divide something in half I would make how many groups? If I divide it in half how many groups is that, Claire?

She asked a second student for the answer instead of providing it for the student who did not respond to the question.

Claire: two

Callie: Two. So I should be putting my eggs in two different groups.

Callie: So can you tell me how much is in each group? Okay, it looks like everybody is done. What did you find? If I want ½ of 18. ½ of 18, How much would it be in each group, Taylor?

Taylor: 9

The teacher used a series of questions during this next teaching episode having the students explain the process in their own way deepening their understanding of the concept.

Callie: How did you know how many groups to use? Is there anything that is written up here in the fraction and the number sentence to help you also? What helps you?

Girl 1: 18

Callie: The eighteen helps you. Okay the eighteen helps me know how many I have all together.

Girl 2: The ½

Callie: Exactly. How does this fraction ½ help you by telling you how many groups it goes into? Thank you for those who are participating and paying attention. How does this ½ tell me, Nina?

Nina: That number that bottom number two. There are going to be two groups.

Callie: Okay, this denominator helped me to know how many groups there are. What if I had to do 1/3 of 18? If I wanted to break you into one third? How many groups would that be? Would you turn and talk?

Callie gave a couple more examples for students to practice figuring out how to determine how many groups are needed to solve a problem of this nature. Later, she wrote down the problems she had students work on: ½ of 18, 1/3 of 18, 1/6 of 18 with students' answers. She asked the students to see if they could find any patterns from the problems and discussed fact family relationships with them.

Callie had students work on a problem with dividing fractional parts in another lesson. The problem was, "If you have seven brownies for four friends how much does each friend get?" She gave each student 7 paper brownies and four paper faces to represent the friends. Students could cut the brownies any way they wanted to find out how to divide seven brownies equally to the four friends. In this next teaching episode, Callie is working with a group of two students as they are trying out ways to distribute the brownies. The two students have been working independently on possible strategies. Notice how she allowed the students to explore and discover the mathematics and gain autonomy and perseverance with her guidance.

Faith: I was thinking that we needed to cut, need to cut them in half. Let's see what it is, no wait, so 2, 4, 8, 10, 12, that's 14.

Ben: So we have 7 and 14 (they are putting pieces on the four friends trying to see if either worked).

Faith: Do you just want to cut them now?

Ben: They each just get two or three (they start cutting the seven brownies). So they each get seven?

Faith: No, they don't each get seven.

Ben: Maybe they all get two or?

Faith: They might all get two.

Callie showed up at their table and wanted to know what they were thinking.

Callie: How many brownies do you have?

Faith: Seven

Callie: Seven brownies. And how many friends do you have?

Faith: Four

Callie: Faith, do you have an idea in mind? I see you cutting.

Faith: Yeah, I'm going to cut them in half and see what they what they each get. One half of a brownie and go back and see how many wholes there is.

Callie: Do you see what she is doing?

Ben: Yeah

Callie: Do you want to watch her first and help her or do you want to cut yours also?

Ben: Cut mine

Faith: So this is 14. Seven cut into 14. (starts to place the pieces)

Callie: Still working on it?

Faith: No. It's not going to work.

Callie: That's interesting. You want to know what I thought was interesting? You both did the exact same thing but you did it in different ways. You cut yours and you counted yours and then drew a picture to go with it. But it doesn't work does it.

Faith: We need two more brownies. Well a whole.

Callie: A whole, exactly because you need two halves. Okay, see what else you can do. Think about other ways that you know how to divide those brownies up.

Faith: Do we…

Callie: Do you need extra brownies. There is an envelope of extra brownies up on the stool with extra brownies.

Callie left the group to allow them to continue exploring divisional conjectures.

Ben: Or we can just use mine

Faith: Yeah, so do we divide them into thirds? That's what will do? Or do you just want me to cut one more?

Ben: No, that doesn't work because one doesn't

Faith: No, I'm going to go like this. I'm going to like this and cut one off of this and it's going to be 14.

Ben: You're just going to do it over. So they each get 1/3rd of one.

Faith: Nine, ten, eleven, twelve, thirteen, fourteen. There's three left so we'll use them.

Ben: So where are those supposed to go?

Faith: These are the brownies.

Ben: So they each get 1/3rd?

Faith: Yeah, I think. We'll have to see what. (They try to distribute them). These are halves.

Ben: I wonder how many we have?

Faith: Let's see. One, two, three, four, five, six, seven, eight. No, yeah, because they are each going to have eight. Two, three, four, five, six, seven, eight.

Ben: We still have one half. Still one half.

Faith: Yeah, but that's going to be for our second time. That's going to be for our second problem.

Ben: Well this is our first.

Callie came back to the group to see what developments had occurred.

Faith: We found a way but we just cut them in half.

Callie: So you cut your halves in half?

Faith: Yeah, we cut a couple in thirds and they each get eight.

Callie: Okay, let's go back and let me understand this. So you took your halves and you cut them in half. So how much is that?

Faith: One half. One whole.

Callie: Okay, So you left your whole in halves? So you only cut it once in half?

Faith: Yeah

Callie: Okay so you have halves. So then what did you do? Did you go back and get more brownies?

Faith: Yes

Callie: Oh, okay so I thought that you cut them in half and then cut them in thirds. So this is a total of seven brownies?

Faith: Eight

Callie: How many whole brownies did you start with? So you started with seven whole brownies?

Faith: Yes

Callie: Okay, and each one of them, so is this one person?

Faith: That's one whole. No, that is all one person.

Callie: This is all one person. So they get this much out of seven brownies?

Faith: Uh hum

Callie: Ben does that make sense to you? If I start with seven brownies and you're saying one person is going to get this which looks like four whole brownies to me.

Faith: Or that could be equal or this could be this could be halves or that it could a whole.

Callie: But we have to remember, Ben are you paying attention, are we starting with seven brownies? Where's your seven brownies? What's all these? Yeah, sweetheart your showing me seven brownies that have already been divided. I need to start with seven whole brownies. Where is your seven whole brownies?

Faith and Callie count the pieces up and start with seven whole brownies. Faith had extra fractional pieces that they took out of the activity.

Callie: So we have one, two, three, four, five, six, seven whole brownies, correct?

Faith: Yes

Callie: Now we have four people. Let's move these all off to the side so we know. Now we have four friends. You've already cut them. Can you somehow give them to four friends? Four friends that get equal. Okay, so you're starting off. So let's stop for one second. So it looks to me then like you are starting off with each person gets one full brownie.

Faith: Yes

Callie: Each person gets one whole brownie and then you have three left. How are you going to divide those three up so that four friends get the same amount?

Faith: They each get one half.

Callie: They each get one half, try it. Here's your other half. Let me think about this for one second. Did you have this one divided into thirds?

Faith: Yes

Callie: Does one third plus one third equal one half?

Faith: A third. Wait no, one third plus one third.

Callie: Ben can you help me out here? One third plus one third does that equal a half?

Ben: No, uhm one third plus one third couldn't be a half because if you put those off there you would have to cut the middle in half. So that would be more.

Callie: But would two thirds be a half?

Ben: NO

Callie: NO. I think you were on the right track. Back up a little. Let's go back to where you had half where each person is that one thirds?

Faith: That's one, that's one, that's one.

Callie: Okay so each person has a whole.

Faith: Then we give each person one half.

Callie: Okay, so can you do that? So now each person has what?

Faith: One and a half

Callie: One and one half. Now what do you have left there?

Faith: Three pieces. Three thirds. Quarters. No

Callie: They're not quarters because quarters are what? You have three thirds left.

Faith: Yeah, we could throw these away.

Callie: That would be great except we won't have our seven brownies to start with (both giggling) remember. That would be super easy but we can't do that. Because we only have seven brownies remember. Ben any ideas about what you're going to do? Because can I give each one a third?

Ben: NO, because one would be stuck without uhm one of those.

Callie: Exactly, and that would not be fair now would it. They wouldn't be equal. Not fair shares.

Ben: We could just throw those away.

Callie: We could but we're not going to waste any food today.

Ben: You could pretend that this is a whole and you put it into four.

Callie: Why would you make it a whole again and why would you divide it into fourths?

Ben: So each person could get a fourth

Callie: What do you think about that idea, Faith?

Callie wanted students to explore, interact, and discover mathematics in all 12 lessons that I observed.

Key issues.

Discovery. Callie did not give an answer to the class for a math problem during my observations. She would ask a question or ask another student to answer. She would then confirm that an answer was correct by repeating the student's answer and then either model the next step or ask qualifying questions to explore mathematical reasoning. She would explore incorrect answers after identifying the reasoning of the student and then led a discussion to facilitate students' thinking to find the correct answer.

It was important to Callie that students discover the mathematics instead of telling or lecturing. She wanted her students to have a different mathematical experience than she had in school,

> When I was taught, I didn't get it. It was a lot of memorizing and just I'm going to tell you what it is and you memorize it. Now, kids love it. They have fun in math. We don't pull out the book and do 1 thru 50 today and move on.

Callie told me later, "I think that students are more accountable to the mathematics if they discover it on their own." She used explicit instruction and modeling as shown in the episodes to guide students through their discoveries. She always made sure that each student could explore the activity on his or her own.

Callie used activities and manipulatives in her teaching so students could touch and manipulate objects to understand abstract ideas. Every activity that I observed had a problem or discovery to find. Her activities were the focus of the lesson and the teaching of mathematics came from interacting with students and the activity. These activities were creative and used elements of surprise and exploration. For example, the paper fraction lesson allowed students to manipulate a piece of paper to find equal fractional parts of "paper brownies." Real brownies were presented at the end of the lesson. Callie asked that each person divide their real brownie into equal parts to share with four friends and was required to discuss their strategies with a partner. This was an extension of the lesson that allowed students to apply concepts and transfer skills they had learned from the lesson.

Multiple activities were conducted in some of the lessons. Callie would break down a lesson into a series of activities that built on each other like the fractional paper activity that developed into using multiple fractional parts to make a whole. Her first activity in that lesson was to have students identify equal parts. Then she conducted an activity to show students how to identify a fractional part of a whole. When she felt they had those skills down she had another activity where students added different fractional parts of a sheet of paper to find multiple ways to use fractions to equal one whole.

Manipulatives were used in every activity I observed. I saw paper brownies, Hershey bars, quilt squares, and pieces of paper used as manipulatives as well as some of the more traditional ones: tiles, dotted paper, pattern blocks and geoboards. The manipulatives and the activities had multiple solutions to the problem. Students shared what they did with the manipulative to arrive at their answers with the class. Callie used manipulatives to allow the students to interact with the mathematics of the lesson. She told me that learning and understanding improved with each student's personal experience with the mathematics.

Turn & Talk. Callie asked students to explain to her their reasoning and procedure for what they were doing continuously in every lesson. She would asks several questions and would reiterate what she thought the student was saying back to the class to make sure that she understood exactly what the student was saying. This allowed her to know what the student knew and did not know and included the rest of the class in the discussion.

Sharing thoughts with each other and asking questions was evident all the time in every segment of her mathematics class.

Students enjoyed the opportunity to "turn and talk" when the teacher directed. I was amazed at how attentive and quiet the students were until they were asked to turn and talk. I saw the smiles come on their faces and their voices racing to share what they thought with their classmates. Students would occasionally not answer a question that the teacher asked in hopes that they could turn and talk with each other before they answered the question. They would be silent to a question until asked to turn and talk and then numerous ideas and thoughts would emerge instantaneously. It presented a way that all the students could be a part of the class. Callie would also use turn and talk if she wanted to know what students were thinking or thought that they might be a little confused. Discussing mathematics was a very important part of Callie's mathematics teaching. She had students share their discoveries and thinking with each other in class as shown in the previous episodes. She told me that this kind of communication allowed students to teach each other and may be more helpful at times than the way she explained it.

Playful "social" environment. Callie used a soft playful voice in the teaching of mathematics in every lesson I observed. She would introduce the activity and model the task as if she were playing and having fun. Callie created a classroom where students felt safe and comfortable to explore mathematics. She was quick to protect students' incorrect thinking and redirect the student in a supportive way. She frequently would justify student's thinking or feelings and reaffirmed their worth.

Students took risks in every activity and lesson I observed. There was no risk for a student to lose points or fail an assignment because I did not see Callie have a regular assignment that students turned in. I was curious about how she could assess the students in mathematics so I randomly selected a student by looking at a name on one of the desks. I spontaneously asked Callie in an interview; if I were Megan's parents, what would you tell me about her mathematics performance. She responded:

> I can tell you that Megan has come a long way in math. She started very incompetent. She was not quite sure of herself. Her communication was weak at the beginning, I have documentation of her problem solving book. She has come from not being able to explain her thinking to now she has pages of detailed thinking. Her number sense has grown. When we worked with the hundreds chart at the beginning of the year she didn't know that if you wanted to add ten more you just go down one row. She's still unsure about sharing in class. She's come a long way but she can continue to improve in that area. She does well in small groups but sharing is hard for her to the whole group.

Callie knew where her students' skills were in mathematics. She had students write in journals and problem solving books, conducted independent activities, had personal discussions, and did calendar math. Observations and discussions with the students allowed Callie to know her students competencies in mathematics. Callie structured her lessons so that students had the opportunity to learn math in a variety of social settings: direct instruction, modeling, trying it out in activities, paired explorations, independent practice and discussions with classmates and the teacher without the feeling of failure.

Modeling. Callie used modeling to give students an example of how to execute a mathematical activity or procedure. This was her way of showing the students how to accomplish the task. She told me when she develops an activity she has to decide when to present a problem or model a procedure during her lesson. Callie noticed in the ½ yellow problem students were struggling to figure out how to find ½ of a set of blocks. She used the overhead projector to walk the students through the process of determining how to find a half. She used questioning, colored blocks for visual manipulation, and manipulatives for students to find proofs and to model procedures students would need to solve and understand the mathematics. Callie also modeled and had students present their own models of their mathematical thinking during my observations. When Callie asked Ben why he said 2/4th as an answer he was able to model his thinking that Callie used to have students explore 2/4 and 1/2. This led to the proof that the design was half yellow. She continually used students' thinking to model the process or show the logical understanding of what to do and why it worked.

Discussions. Callie used discussions to guide students through their mathematical discoveries. She used them to explore, explain, instruct, prove, discover, and experience mathematics. She would ask questions, have students turn and talk, independently interact with the task and had students share their discoveries with each other and the class. Callie used discussions to model and guide students' thinking, to inform her about student understanding, and to allow students to socially interact with others in their construction of mathematical understanding.

Callie revoiced students' contributions and asked questions to verify that she understood precisely what they were saying. Students felt their contributions were important whether they were correct or not. The students were equal shareholders during the discussions as they shared their discoveries, thinking, and examples with the class. Callie used discussion to teach mathematics by asking questions and developing understanding. She would frequently ask why did you do that? What do you mean? If it isn't what do you think it is? If I make it like this does that make it a ½ design? Why?

Why does that work? How do you know? If I wanted to divide it by 1/8th how many groups would I have?

Callie used discussion to clarify mathematical operations and understandings. She guided students on how to execute the procedure to find the answer. Discussion was used by Callie to show students multiple examples in finding how many groups you would use to get the answer to a specific kind of problem. She talked through the process to guide students through the steps needed and would break down the process into specific skills. Callie used a step by step discussion to go from ½ of the class to solving problems that asked for a fractional part of a whole number. There were 18 gummi bears and the students were to find 1/3 of 18. This problem was presented after she modeled ½ of 18.

> Callie: What if I wanted to find 1/3 of 18? So when I look at the fraction 1/3 of something it was one out of three. I remember the bottom number, the denominator represents the whole, right. So if I have three and I take my 18 and put it in groups of three I'm dividing 18 into groups of three because 1/3rd tells me that. Can everyone see that 1/3rd where I'm getting that three?

In this case, Callie started by discussing what students had experience with, dividing the class in half. Then she determined that students were having difficulty determining how many groups to divide the number by. Callie discussed finding the fractional part of a number using 1/2, 1/3, and 1/6 of the same number to provide guided practice to finding how many groups to divide the number into. She used representation to show students how the number could be divided. The class talked about the different ways they came up with the answers and Callie verified the legitimacy of the conjecture.

Chapter 4

Findings

It is important to note that substantial time has passed between the design of the study (Nov. 2007), the collection of data (April-May 2008) and the analysis and writing of the analysis (Jan.- Oct. 2012). During this period of time mathematics literature and educational environments have evolved. The findings are presented here from the time period in which the phases were enacted with attempts to relate those findings to current perceptions in literature when possible. I determined meaning of the data by focusing on the case itself and to the grand question. I tried to understand behavior, issues, and context with regard to this particular case. I explored the meaning of the case through “categorical aggregation” and “direct interpretation” (Stake, 1995). I used these strategies to reach new meanings about the case through aggregation of instances until something could be said about them as a class. I searched for meaning by looking for patterns, for consistency within certain conditions which Stake calls “correspondence.” I analyzed the case by conducting a cross-case analysis by searching for correspondence of the three teachers’ teaching collectively using their key issues (Table 5) and teaching episodes I observed in their classrooms. I included direct interpretation of meaning of a unique aspect of the case from one individual teacher’s teaching. I believed the direct interpretation that I found in only one of the teachers was a necessary dimension to the understanding of the case so I decided to include it as part of the analysis. This led to the formation of a set of dimensions of excellent elementary mathematics teaching from this case study.

Table 5

Key Issues of Teachers

Sue Johnson	Kathy Freeman	Callie Hendricks
DIALOGUE * group work * wondering	MATH EXPRESSION * articulate * understand process	DISCOVERY * questioning * activities * manipulatives
LOGICAL SENSE * 4 % = reasonable * deep understanding	IMMEDIATE FEEDBACK * time-outs * white boards	TURN & TALK * social * dialogue * share ideas & thoughts
REPRESENTATION “SEE IT” * math language * interaction w/ it	ACTIVE INTERACTION * games * facts in context	PLAYFUL SOCIAL * soft voice * safe environment * no risk grading
EXPLORATION EXTENSION * Pepsi Problem * trapezoid problem * circumference problem	STUDENT PROCESSING * see and hear * questioning * student driven	MODELING * ½ yellow problem * overhead * manipulatives
FEEDBACK GUIDES INSTRUCTION * “I divert” * spontaneous * homework	MULTIPLE WAYS * PPPW mats * people are different * 3 ways to solve	DISCUSSION * dialogue w/ class, students, and teacher * clarify * assess * guide

I used the teachers’ interviews, observations, and my field notes to describe how these teachers taught in their classrooms individually that led to a set of individual key issues for each teacher in Chapter 3. I suspended the use and exploration of the literature during this process so as not to influence the integrity of the data. In this chapter, I re-introduced literature as an additional source of information to compare theory with the particulars of the case. A combination of the cross-case analysis and direct interpretation blended with research literature resulted in a set of dimensions of excellence of the case. Table 5 is a summary of the three teachers' key issues from Chapter 3 and was used in the formation of categories in the cross-case analysis.

Cross-case Analysis

I used the term teachers in this chapter instead of the term participants. I focused on the "correspondence" of the three teachers collectively using the key issues identified for each teacher in chapter three although deeper analyses of what the key issues mean were explored. I found that, collectively, the teachers were strikingly similar and their teaching philosophies and pedagogical decisions were relatively the same and grounded in research. These teachers created opportunities for the students to learn the cognitive knowledge and skills that will help them gain mathematical power. Mathematical power as defined by *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) "is the individual's ability to reason logically, make conjectures, and communicate effectively about mathematics."

The findings of the cross-case analysis are represented in five individual categories of correspondence but it is important to realize that they were not isolated happenings in the teachers' teaching. The cross-case analysis looked for correspondence of all three teachers' teaching collectively and was presented here in categories. They are not separate entities but are integrated together in the teachers' teaching of mathematics simultaneously. These categories were identified by evidence of similar occurrences in all three classrooms. The categories are not presented in any sequential or hierarchical way. I found that the data showed few differences among the teachers and were identified in the direct interpretation section or in each category individually.

Category 1: Limited reliance on textbook. When math class started I expected students to get out their textbook and open it to a specific page to get ready for instruction. That never happened in any of these teachers' classrooms throughout my observations. Students thinking, actions, and dialogue took the place of the textbook. Sue used a "resource book" to check for measurements and forgotten algorithms and was used only once in my observations for a measurement conversion. Kathy used a workbook worksheet at the end of the lesson to check for student skill. She told me that she uses worksheets because her students would not be able to successfully complete assessments in the first grade unless they had experience with the structure of the problem as it is presented in various ways on the page. Callie had the math books collected on the top shelf of her cupboard and said she gets them out "a couple times a year to use the problems to check for understanding." I did not observe her using the textbook in any of my observations.

Current research suggests that teachers use curricular materials differently in the designing of their intended lesson. McCrory et al. (2008) surveyed 63 college instructors of courses for prospective elementary teachers and found variety in their goals and enactment of the written curriculum. Remillard (1999, 2005) and Stein, Remillard, and Smith (2007) found phases of teacher usage from written curriculum (printed pages) to intended curriculum (teacher's plan for instruction) to enacted curriculum (actual implementation in class). Although data was not collected on how the teachers used these curricular materials it was evident that the teachers adapted the curriculum uniquely in their enacted phase of teaching their schools curriculum. Remillard (2000) refers to this approach to designing instruction as the curricula speaking to as opposed to speaking through the teacher. Stein et al. (2007) identified three factors used to examine adjustments teachers made between the written and enacted curriculum: beliefs and knowledge, orientation toward the curriculum, and professional identity. Even though specific study of the teachers' use of their curriculum was not studied evidence exists between Stein's three factors. Callie's orientation to written curriculum was clear. Sue and Callie talked about their analysis of the textbooks and their discomfort of the written materials design and presentation. It was clear that all three teachers obtained public professional identities in their schools and communities. Lo, Kim and McCrory (2008) found that the instructor's beliefs and the goals that they set up for their class shaped their decisions about how they used the curriculum.

Textbooks were not a pedagogical match for the way these teachers taught mathematics. The teachers wanted the students to interact with the mathematics using problem solving techniques. For instance, Sue created real world problems and asked students to find solutions, Kathy asked students to determine which strategies they could use to solve problems, and Callie enabled students to collectively try out conjectures as they interacted with problems. Textbooks are usually designed to show an algorithm or procedure to use and a collection of problems to practice the algorithm. These teachers wanted the students to conceptually understand the mathematics of what they were learning and textbooks are typically used as a procedural pedagogical device. Unlike routine textbook questions, non-routine problems and open-ended questions are deemed to offer opportunities for students to engage in situations that may require them to formulate hypotheses, explain mathematical situations, create new related problems, and make generalizations (Lajoie, 1995; Silver & Kenney, 1995; Stenmark, 1989). The teachers wanted students to experience the mathematics before they explored the algorithm. “I want them to figure it out by doing some activities that we do first,” said Sue. Two of the teachers talked about the limitations of textbooks in explaining their reasoning for not using them. Sue did not always like the sequence of events or how the text taught some concepts or procedures. Callie did not want her students to sit and do “problems one through fifty and move on.” She despised the use of textbooks in reading, as well, because textbooks, “do not address the specific needs of individual students.”

All three teachers worried that the textbook’s reading level got in the way of students’ understanding of the material. Sue said, “This book has a lot of reading in it and sometimes I think that some of the explanation here could be confusing.” Bruning, Schraw, Norby, and Ronning (2004) found that when students encounter a word problem, they first need to make sense of the problem’s text. The schemata for text comprehension generate a second set of schemata for mathematics problems that lead to problem solutions. The teachers felt that the textbook limited their ability to teach the students and that they could teach mathematics better without it. Each teacher told me that they understood what they were teaching well enough that they did not need a textbook to teach it.

Category 2: Encourages classroom dialogue. Students in these classrooms were not sitting quietly in their own seat during mathematics class. They usually were in groups, positioned closely on the floor or conducting activities to discuss and experience math. Students were talking and interacting with math problems continuously. These teachers used dialogue to help their students develop habits of thinking, analyzing and verbalizing their own and others' reasoning and allowed students to actively discuss their thinking with each other as they attempted to solve problems. Researchers increasingly consider the quality of classroom discourse to be one of the most critical elements in effective schooling (e.g., Calfee, Dunlap, & Wat, 1994; Chinn, Anderson, & Waggoner, 2001; Kuhn, Shaw, & Felton, 1997; Nystrand & Gamoran, 1991; Wiencek & O'Flahavan, 1994).

Group work was one method of discourse used by these teachers to have students experience mathematics. According to Lampert (1990), the content of mathematics lessons should be in the arguments and dialogue in the classroom. Sue had at least one group work segment for each day's lesson but students usually would get into their groups two or three times. Her groups were organized with three students in each group and changed the members of the groups only for one day during my observations. Students discussed their solutions and explained how they got their answers. Sue told me that students usually figured out what they did wrong when they had to tell someone else what they did and why. They were taught to agree and disagree and talk about how they got their answer. Students would also get into groups to conjecture solutions and discuss possible strategies to solve a unique problem, most of the time without directions or algorithms given by the teacher. "You have to really listen to kids because a lot of times what you think you said and what they understood are different," says Sue. Kathy assembled students in groups for the games portion of the class. Students socially discussed what they were doing and counting in the context of playing a game. These groups had either two or three students in each group and changed frequently. "Putting it into words and being able to verbalize it is vastly different from just being able to do it," says Kathy. Callie also put students into groups of two or three students to attempt to solve problems intuitively and to talk about each others ideas and conjectures for solving them. Callie explained,

> By letting them turn and talk in groups, turning and talking to a partner, helps them also with asking and answering questions because they know it's not just them but they'll have that chance to have that interaction and dialogue before answering. To also clarify anything if they need it and it helps take the risk of failure or putting yourself out there off of them. When they turn and talk they all have great ideas but they just don't always want to share them and it gives everybody a chance to talk and hear one another versus just a few who raised their hands.

Group work was not the only time that discourse was occurring in these classrooms. The teachers engaged students in dialogue of mathematics at all times whether it be group work, whole class discussions, problem solving tasks, independent practice, or homework. The teachers used dialogue with students to explore and develop how students were conceptualizing mathematical ideas. Cobb, Yackel, and Wood (1990) found that, "Opportunities for students to construct knowledge arise as they interact with both the teacher and their peers." Callie suggested, "Sometimes students can understand another student's explanation better than mine." These teachers did not lecture and tell students what to do. They asked questions that created discussions about how students were understanding and interacting with the mathematics. They actively monitored and facilitated the dialogue around mathematical knowledge. Chazan and Ball (1995), found allowing students to discuss their thinking about a problem with one another fails to promote a strong mathematics community without the important role of the teacher to "facilitate conversations about particular ideas, but also to help students understand misconceptions." According to Lampert (1990), students should make conjectures, and communicate solutions, try to convince themselves and one another of the validity of particular solutions or answers, and rely on mathematical proof and argument to determine the validity of answers. Instead of telling students whether they were correct or not, these teachers would create cognitive conflict arguments and questions to probe further thinking and developed deeper understanding of the mathematical concept and/or procedure. When the teachers extended a problem in their teaching is one example of how they created cognitive conflict to deepen understanding.

Dialogue was also used to inform the teachers about how students were processing the task and choosing to solve a problem, identifying and externalizing student errors and highlighting unique mathematical thinking. Teachers can use information gleaned from student talk to inform their instructional decision-making practices, including problems to pose and follow-up questions to ask (Franke, Fennema, & Carpenter, 1997). At various times the teachers would stop group activities and redirect the students through whole class discussions using a groups' thinking, conjectures, and logic to connect the mathematics to the problem, to extend students' thinking about how they were attempting to solve the problem, or having students transfer mathematical concepts and procedures to another context. This informed students on how they were processing and connecting the mathematics. Students were required to explain their thinking and how they chose to solve a problem. Callie said, "They know that most questions they're going to have to give me a reason why, they have to explain it." Kathy added, "If they don't talk you don't know what they know. If they don't talk and explain it then they probably don't know it. So I see talking math and understanding go hand in hand." Maher and Martino (1996) found that, Students need to be able to justify and explain ideas in order to make their reasoning clear, hone their reasoning skills, and improve their conceptual understanding. All of the teachers told me that if a student cannot verbalize the mathematics then they really do not understand it.

Category 3: Focus on conceptual understanding. The teachers in this study made pedagogical decisions specifically for the purpose of developing students' conceptual understanding. They chose not to use a textbook because that instrument usually did not foster understanding of how the mathematics works and why. A problem solving format with dialogue and discussion to interact with the mathematics to develop and extend concepts was used by the teachers. They made connections of the mathematics using representations and manipulatives and expected students to discover ways to solve problems and build personal conjectures. The environments that the teachers created engaged students in discovering the mathematics for themselves and revealed those discoveries to others through discussions and questioning techniques. The teachers used facilitative and guiding teaching strategies to help students make sense of the mathematics and direct their thinking more deeply.

A major focus of the teachers for students was to develop their understandings about the mathematics they were teaching. First, they had to understand it before they could teach mathematics for understanding. The National Mathematics Advisory Panel (2008) states that "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching." Ball et al. (2005) argued that, "effective teaching entails knowledge of mathematics that is above and beyond what a mathematically literate adult learns." Callie said, "I would have to be able to understand it before I would be able to ask questions or before I would be able to understand their thinking." Sue said, "I have a better understanding of some concepts and can delve into them deeper because of my own understanding to present it to the students to help them understand it. If my understanding wasn't clear then I don't present it to the students in a clear way." The understanding that the teachers taught students started with students current understandings. Kathy said,

> If you just plan a lesson and you do it just like the guide, then it really doesn't matter if the kids understand it or not. You just teach it. But that's not the way of teaching math. You have to see where they are when you start, what they're grasping from it, where you need to go from there to get them to understanding and you have to make all the adjustments that you need.

The National Research Council's (NRC) research project called Adding it up: Helping children learn mathematics (2001) suggests that the problem solving approaches that young children bring to the learning of mathematics must be nurtured and built on, not extinguished. Understanding for these teachers was exploring the mathematical journey and making connections along the way. The focus of their teaching was in the process of understanding the mathematics on the way to solving a problem and finding solutions. Instead of giving the students a task and measuring how many correct responses there were, these teachers gave students a task and observed how much and what kind of information was needed in order to complete the task successfully. They wanted their students to be able to use the information they were learning in real situations and different contexts. Callie said,

> It's not the learning part, it's the understanding and being able to transfer it to something else. I can teach them five times three equals fifteen but that's pointless if they can't use that information and understand that information and apply it to problem solving. It's about discovery and understanding that it means groups and manipulating objects and visualizing it and doing that over and over in all different ways using cubes, graph paper, objects in the room, ourselves, skittles, to discover and learn and understand it.

The National Research Council (2001) found that, “Procedures can be taught essentially by rote, of course. The challenge for all teachers is to help students develop the conceptual web of information and metacognitive knowledge underlying the procedures and strategies for using them flexibly” (p. 334). The teachers wanted their students to learn mathematics to use in their lives for the rest of their lives. They taught for understanding through making sense of how the mathematics works, why it works, when to use it, and what it was. The teachers improved students’ conceptual understanding by creating experiences for students to conduct deeper processing and higher order thinking. They actively engaged students’ to experience mathematics, and used students’ thinking to develop concepts and procedures.

These teachers wanted students to learn mathematics with higher level thinking. Using Bloom’s Taxonomy as an example, these teachers asked students to learn mathematics through the higher levels of cognitive thinking about mathematics. Kathy said, “I don’t want just rote learning. Conceptualizing something to me equates to understanding. Just regurgitating information or doing it on the piece of paper doesn’t show me that same understanding. I want them to understand. I need to hear what their thinking is.” The teachers in this study wanted students to analyze, synthesize and evaluate the processes and concepts that were being developed in their minds. Sue said,

> I think over the years that we’ve taught algorithms and formulas but we didn’t have an understanding of it. What am I really doing here? Why are we doing that? Why does that happen? When doing two digit multiplication, why do we put a zero on the second line? Why do you put it there instead of saying put a zero there. But why do you have to do that?

The National Research Council (2001) found, “There is growing evidence that students learn best when they are presented with academically challenging work that focuses on sense making and problem solving as well as skill building (p. 335). The teachers felt that if the students learned the why, the when, the how, and the different strategies you could use to find answers you would have a deeper understanding of the mathematics. “They [students] learn by counting but they don’t have a real sense of what that is and then you give it meaning by counting objects and letting them see what it looks like and they learn by exploring their world,” says Kathy. The teachers wanted the students to metacognitively understand how they were processing mathematical tasks. “You know they can add the three numbers together but do they really truly understand what strategies they’re using and how helpful those strategies are going to be along the way as far as knowing the basic facts. Can they put into words what their brain is doing?” proclaimed Kathy. The teachers listened to student responses to assess what the students needed to develop their understanding of a concept or procedure. “If I think they are not understanding it, I slow down or take more time or try and find a different way to explain it or maybe do more practice problems or try to find some hands-on things so they can understand,” says Sue.

These teachers actively engaged students in the discovery and exploration of the mathematics they were teaching to enhance understanding. This was accomplished by using a problem solving pedagogical strategy. The teachers would create a problem or situation and asked students to solve it. “If I show you how to do it and let you work with it, you might understand it,” says Callie. Students were interacting with the problem and themselves as they created conjectures to solve a problem. According to Davis, Maher, and Noddings (1990), Students must experience the concepts in order to build understanding. The teachers conducted whole class discussions, time-outs, and used visual representations at opportune times in lessons to connect the mathematics they wanted students to learn to the way the students were working with the problem. They often asked questions that extended the problem to give students additional experience with the concept or procedure to deepen the understanding of what they were learning. Brooks and Brooks (1993) suggested that when students construct the process required to solve problems, rather than having it done for them, they learn to make sense of information.

The teachers carefully selected problems that students would be interested in and involved their personal life experiences. The Pepsi problem, fencing around the farm problem, candy fraction problems, measuring circles, windows and floors for area, etc. are some of the examples used by the teachers. The National Research Council (2001) found that, “The conceptual basis for operations with numbers and how those operations relate to real situations should be a major focus of the curriculum. Students should encounter a wide range of situations in which operations are used” (p. 413). In these classrooms, the teacher, students, and the task interacted in dynamic ways to develop students’ thinking and learning. The teachers posed questions and problems, assessed student assumptions, preconceptions and performance, actively and consistently supported the students’ cognitive development as students tested their own ideas and thinking with more reliable observations and data, learning to extend their own understanding of the mathematical concept and process being taught.

The teachers used representations and manipulatives to help students interact and explore mathematical concepts and procedures. “You realize the benefits of kids manipulating things to understand math,” says Callie. “I probably like to draw pictures to kind of show a process of the whole," says Sue. "I use manipulatives to give children a perceptual understanding of mathematics," says Kathy. The representation or manipulative allowed the student to interact with the mathematics that was being taught. They were used by the teachers to interact with the students about the mathematics and to externalize what students were understanding and doing in their mind. Carpenter and Fennema (1992) found that manipulative materials are of major use in helping teachers understand what the child is thinking. The teachers used a manipulative or representation to show students relationships and connections to concretely see what was occurring abstractly with the mathematics. Heibert and Carpenter (1992) noted that when students internally make sense of the concepts and then make appropriate connections with the external representations, understanding has occurred. The pedagogical decision by these teachers to use manipulatives and representations benefited the students by allowing them to interact and discover mathematics and helped the teachers understand what the students were thinking.

These teachers taught the students with the curriculum instead of teaching the curriculum to the students. The lessons developed spontaneously from information the teachers were receiving on how students were thinking and performing. Kathy stated, "That's why it is so important for me to see kids solve problems. So I can see how they are thinking, what they are thinking. I've got to know what they are thinking all the time." The teachers were always asking questions and assessing students' performance and thinking. They wanted to know what students knew and what they did not know. Kathy stated,

> You have to know what it is they are not understanding. You have to know which part to tweak and what part you haven't taught well so you can go back there and

> work on that. You have to know what they're taking in so you know how to say it differently if they are not getting it.

These teachers used dialogue and questions to explore what students were thinking about concepts and procedures. "It's like yesterday when the one group said they did not think it was logical. Then I just kept asking questions because I think if you ask questions you get kids to keep thinking about what to do next," said Sue. According to Fosnot (1989), in order to maximize students' understanding, it is important to probe students' thinking to cause them to ask questions of themselves. These teachers guided students to process their own understandings of the mathematics. These interactions, interpretations, and actions of the teachers and students determined what the enacted lesson became. Their attention to and responses of the students shaped how the lesson progressed. These teachers wanted students to develop conceptual, procedural, and relational understandings of mathematics. They wanted students to conceptually make sense of what it was they were doing, how to procedurally execute the concept or algorithm in a problem context, and how to transfer and connect this knowledge to new situations and other mathematics.

Category 4: Used experiential problem solving activities. These teachers chose to teach mathematics differently than their peers at their respective schools. They chose to teach using experiential instruction while most of their peers used conventional instruction. The NRC (2001) defines conventional instruction that "tends to be rule based with emphasis on helping students become quick and accurate in executing written procedures by following rules" (p. 240). Experiential instruction tends to be problem based with the focus on explanation and understanding. The emphasis in experimental instruction is on why a procedure works rather than treating it as a sequence of steps to be memorized. Teaching this way is more difficult and takes much more preparation to teach. "It's a lot more work preparing and getting materials ready and that's usually the negative part of it," says Callie. Good problem solving tasks have been identified as having common features (Stein, Grover, & Henningsen, 1996) that each of these teachers exhibited. First, they are accessible to a wide range of students yet have no quick solution. Second, they require some amount of investigation or data gathering. Third, there are multiple mathematical paths to a solution or solutions. Fourth, they present opportunities for generalizations to be formed about mathematical relationships. Fifth, they require problem solvers to justify their steps and conclusions based on the givens. And, lastly, they allow for sense-making in that solutions and generalizations can be understood by reference to the original problem context. The teachers talked about being prepared and thinking through the lesson with possible questions they may have to ask and to be ready for the questions that the students might ask. They had to spend extra time in their planning of lessons by developing problems, preparing materials and manipulatives. They also talked about how other teachers resented or did not support this way of teaching. So, why did they choose to teach this way?

One reason the teachers chose this pedagogical strategy was because they wanted to benefit the students. "If I just stay in the basal and just did paper and pencil I would not be improving, I wouldn't be doing any justice for the kids," says Callie. She continued, "When you have no basal then your teaching focus is on the kids and their needs." Research has shown that using problem solving context should be central to school mathematics (Bruning et al., 2004; NCTM 2000, 1991; NRC, 2001).

Research also shows that using a problem solving context to mathematics instruction helps students apply their knowledge to a variety of other situations and problems. "It's an opportunity to use problem solving and to see some of the connections between numbers and applying it to life," says Sue. These teachers developed math problems that applied to students' lives and what they were interested in. "I have had students say that I'm going to graduate from high school and I'm going to farm so why should I learn this. You know even if you are going to farm you need to know math and this is going to help you. They can make the connection to wanting to know this," says Sue. Using proportions to find the best price of pop and figuring fractions by sharing seven brownies with four friends are some of the problems the teachers created for students to explore mathematical concepts and procedures. Chipman (1988) and Jones, Krouse, Feorene, and Saferstein (1985), provide evidence that using different types of problems in clusters with adequate relational connections among problems significantly improved student problem solving skills. These skills also better prepare students for the mathematics that they will experience in future mathematical studies they will encounter in the coming years. "Recalling facts and rote memorization might score them well on a test but it doesn't mean they understand it. They won't be able to apply it to something else and they won't be able to carry it on year after year and be able to apply it," says Callie.

These teachers did not want to teach mathematics the way they were taught. They felt that there was a better way to teach mathematics than the way they experienced it when they learned it in school. "Because the way I was taught, I didn't get it. It was a lot of memorizing and I'm going to tell you what it is and you memorize it and I never really understood it, says Callie. Sue exclaimed, “I’m trying to get them to understand because of the frustration or the misunderstanding that I had of something. I don’t see my students dreading math.”

Other ways students benefitted from these teachers’ choice to teach mathematics through solving problems are many. The teachers wanted the students to actively do the math. “Kids are so much more involved and they really understand it versus me telling them something. Them discovering it makes them accountable and want to learn more,” says Callie. The teachers set up instruction so that each individual student took part in the interaction with the mathematics. They had students work problems and show their thinking on individual boards, gave each student a set of manipulatives to work with, and asked each student to determine their own path to solving the problem. The National Research Council (2001) found that, “Problem solving ability is enhanced when students have opportunities to solve problems themselves and to see problems being solved. Problem solving can provide the site for learning new concepts and for practicing learned skills” (p. 66). The students and the teachers liked teaching and learning mathematics in a problem format where students discovered and teachers facilitated the learning. Good and Brophy (2000) found, “Students usually enjoy responding actively rather than merely listening; opportunities to interact with their peers; situations that invite thought by posing divergent questions: and activities with game-like features, such as puzzles and brainteasers” (p. 19). This pedagogical format also benefitted students because it helped students learn the skills that the world is looking for in the future. Senge et al. (2000) found, “Employers of tomorrow likely will place a much higher value on listening and communication skills, on collaborative learning capabilities, and on critical thinking and systems thinking skills – because most work is increasingly interdependent, dynamic, and global" (p. 51).

These teachers used open-ended problems that had multiple solutions. Students were asked to find solutions to the problem and developed their own conjectures to solve them. The experience taught students that mathematics is a process of thinking actively and not only about an end product. The teachers presented the whole problem and guided the students to break the problems into parts so that they made connections between the parts and whole. This taught students how to gather, apply, analyze, and evaluate information to a concept or procedure instead of following a series of steps to arrive at an answer. This led to students discovering unique algorithms to use to find proofs for problems. "If you listen to these kids there are ten different ways they solve a problem and that's great," says Kathy. The teachers wanted students to learn that mathematics has multiple ways to find solutions to problems. Kathy added,

> I love to see all the different ways they solve problems and they have to talk about it. I guess my goal is not only seeing what their thinking is but for them to understand that you just don't have to do it one way.

All the teachers focused on multiple pathways to solving problems as long as the algorithm was examined and proven to be effective in other examples. They felt that students think and solve problems differently so by teaching and emphasizing multiple solutions to problems more students could be successful in mathematics. "These kids have to know there's lots of different ways so they can find a way that works for them," says Kathy. "There's lots of different ways to solve problems and when you hear kids when they explain their thinking there is so many times they'll come up with things that I would have never thought of," says Callie. She added, "They can solve it the way that they know how to solve it that makes sense to them versus what somebody's telling them to do." These teachers' students learned to build meaning and describe their thinking through their actions and interactions with the problems and the class. The problems allowed students to test their current understandings of mathematical concepts and procedures helping them make sense of and retain the mathematics they were learning. Students experienced mathematics by making connections to real world experiences and helping students apply that knowledge to other situations showing the value of the discipline.

The teachers in this study wanted students to experience and *experiment* with the mathematics. They had students invent their own conjectures by not presenting algorithms at the beginning of a problem. Kamii (1989) stated that when children invent their own algorithms they are more competent with it than if it was told to them. She added that, "procedures children invent are rooted in the depth of their intuition and their natural ways of thinking" (p. 14). Students had to use their prior knowledge and current understandings in formulating their ideas and conjectures allowing the teacher to work with the students and the mathematics at their current level of understanding. In Schon's (1987) view, "Students learn when they act and are helped to think about their actions. Learning by doing forces students to make judgments; reflection helps them recognize their assumptions and see what's important" (p. 202). This student approach to teaching mathematics was important to the teachers. Callie said that when she taught conventionally in the beginning of her career students did not learn a rule by rote. She said, "They didn't learn it because they never got that chance to work with it. I was telling them what rounding was but they never got to experience what it meant. I told them the rule instead of letting them observe it and discover it on their own." The circumference problem where Sue asked the students in groups to come up with conjectures for "pi" of a circle and then tested three of the classes responses to see which one is right or what they would have to change to make them correct is an example of how these teachers wanted students to experience and experiment with the mathematics in a problem context. The students discovered the formula by experimenting with items in the classroom that were circles and found proof for their conjectures.

Category 5: Continually assessing students. These teachers were always moving around the classroom interacting with, observing, and asking questions of students to assess how they were processing the tasks that were given. Assessment of students' understandings, procedures, and logical thinking occurred continuously throughout every lesson. Kathy stated, "I have to walk around and see what they're doing and ask them questions. I'm continuously assessing them and then I know what to teach. That's why we have our time-outs and we huddle." These informal authentic assessments focused on students verbal and physical actions and were interwoven into the lesson. Carpenter and Fennema (1991) found teachers learn more about what students know by observing students' behavior. The teachers assessed more of what students' understandings were rather than how many a student scored correctly. Stiggins (2007) stated, "assessing the quality of mathematical learning remains elusive, and formative assessment has not delivered the promised improvements" (p. 24). The teachers in this study seemed to deemphasize giving students a specific score or grade and emphasized a process of continual development of mathematical understanding in an attempt to build positive self efficacy in their students about mathematics. Senge et al. (2000) suggested, "Young children who get "Cs" and "Ds" on their first math test are very likely to conclude that not only their answers are wrong but they are wrong" (p. 36). These teachers provided a natural environment where students could grow and evolve without labeling them. Senge added, "We need assessments that are designed for learning, not assessments that are used for blaming, ranking, and certifying."

The teachers used informal assessments within the lesson to make adjustments to meet the needs of the students. Kathy explained how she analyzes student understanding,

> I have to first look at two things, I look at what I hear and what I see. Sometimes what I hear tells me that they understand but what they show me tells me they really don't and vice versa. I can sometimes see it on their boards that it looks like they understand it but if they can't articulate it and articulate it correctly then they really understand it. I really need to see *and* hear how they solve a problem.

Studies show that when teachers learn to see and hear students' work during a lesson and use that information to shape their instruction, their instruction becomes clearer, more focused, and more effective (Fennema et al., 1996; Thompson & Briars, 1989).

The teachers used informal assessments to try and externalize how students were processing the concepts and procedures. This information informed the teachers what to do next to progress their students. Kathy continues, "I try to get in their heads and see what their thinking so that helps me know what other examples to give or what to do the next day or how to address the same concept differently so I can reach all of them." The teachers took the time to make sure they understood exactly what a students' response to a question was. Sue added, "I go around as we're doing something and keep questioning enough to make sure that they really understand what I said or to make sure they are having an understanding of the procedures to the process." The National Council of Teachers of Mathematics' Principles and Standards (2000) state that assessment should enhance student learning, help teachers make instructional decisions, and should use many techniques. The document added, “Many assessment techniques can be used by mathematics teachers, including open-ended questions, constructed-response tasks, selected-response items, performance tasks, observations, conversations, journals, and portfolios” (p. 23). These teachers used all of these types of authentic informal assessments of their students continuously throughout every lesson that enhanced students learning of mathematical concepts and procedures.

All the teachers used open-ended questions of the students to determine how they are thinking about the concept or procedure. Callie stated, “Questioning is so important in math and reasoning and explaining your reasoning. If I ask kids questions that really helps me assess whether they’re understanding it or not.” The teachers also analyzed students’ questions to learn about what or how students were thinking. Callie added, “They [students] ask me questions and I can see how deep those questions are or what they are thinking to ask those kinds of questions.” This informal assessment data informed the teachers about what questions to ask next. “What kinds of questions can I ask them to clarify their thinking and their reasoning?” says Callie. “The questions are more open-ended where kids have to think to answer them and is always followed up with how do you know and why,” she added. The teachers also assessed how students were understanding the mathematics by observing how they were interacting with the manipulatives or representations that were presented. Callie was assessing how the students were thinking as she interacted with a couple of students about the sharing seven brownies with four friends problem by seeing what they were doing with the manipulatives and was able to ask the appropriate questions to understand what they were doing and how to redirect their thinking. Verbal and visual feedback went back and forth between the teacher and the students informing both about misconceptions, misunderstandings, and understandings of the mathematics they were exploring. This allowed teachers *and* students opportunities to assess performance of mathematical understandings and skill. Sue stated,

> Through feedback you find out what misconceptions students have. You know who understands it and who is not understanding it. That is why I want students to show their work because a lot of times in there work is where I see where they are making their mistake.

Each teacher's formative assessments differed. Sue had additional formal assessment through daily homework that students practiced executing the skills taught in the day's lesson. Sue frequently changed the homework assignment based on how far the class got in the lesson and whether she felt the students were ready for the problem or not. "I had to modify my assignment because they hadn't worked with decimals because decimals is the next unit," says Sue. She graded the homework and discussed it with the class the next day. Sue added, "I put them in groups to talk about their answers and discuss their answers and hopefully through that process they verbalize and know whether they understand it or not. I kind of go around the room and that helps me to know whether or not they misunderstood something." She also went over problems that the class struggled with to give feedback on procedural or conceptual deficiencies. Students completed quizzes and tests at the end of units and chapters that followed the same format.

Kathy had periodic timed tests, daily worksheets, and chapter tests that informed her about how students were solving problems and building skill. I did not see her share those results with her students during math lessons but they had to be recorded on computer spreadsheets for the district and parent information. They seemed to be used by Kathy to meet the school districts mandates for mathematics instruction. Kathy said that students must get 80% or better on the chapter test or they have to reteach and retest. She said she has a retired teacher work with her and those students that do not show proficiency on the assessment. Kathy stated,

> If they've shown me in their daily work that they really don't have a grasp of what they need to know for the test then I'll take a day out and we'll go over it. It might be just tweaking something or a tiny piece they don't understand we can do the same day as the test. It depends on the magnitude of the tweak.

She added that 95% of the students meet the objectives each year.

Callie did not conduct any formal assessment during my observations. She used multiple methods of informal assessment during her lessons. She told me that her class does math four or five times a day with calendar math, journal writing, etc. but I only observed her mathematics block. She used these additional experiences with mathematical concepts to understand her students' conceptual and procedural abilities in her head. She used the types of informal assessment that all three teachers used. Callie talked about her assessment process,

> During the lesson by asking them questions, observations, and walking around listening to their discussions. Afterwards, because we don't do a lot of worksheet answers type things it would be to look over their work and asking next day questions. Once in a while we throw in some individual work where they would have to then show what they've learned. I can pretty much tell who's getting it and who's not.

When I asked her about her assessment of a random student I chose in her classroom, she demonstrated that she knew her students abilities well and what she needed to do to develop their mathematical knowledge and skill.

Direct Interpretation

The unique teaching strategy of interwoven conceptual and procedural mathematical knowledge and skill of Sue Johnson is so important to the significance of the overall understanding of this case that it needed to be included in the analysis. The National Research Council (2001) determined that successful mathematics teaching and learning included five interdependent and interwoven strands of mathematical proficiency. These include: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. "Mathematical proficiency is not a one-dimensional trait, and it cannot be achieved by focusing on just one or two of these strands." Sue's mathematics teaching showed a clearer example of the connected and interwoven nature of the mathematical strands in her teaching than the other teachers.

Sue's mathematics teaching was contextually different from a procedural or conceptual teaching perspective. An example of the series of events that would occur for students in a procedural pedagogical classroom is different than the experience that students received in Sue's classroom. In a procedural classroom, students would be presented an example, formula, or procedure and have an example of the teacher working through the problem. Next, the students would complete a set of problems, check the problems, and received feedback on how well they performed on the problems. The focus of this mathematics teaching is on the ability to execute mathematical algorithms effectively and efficiently. In a conceptual classroom, the teacher would give students a problem and allow students to explore and experience the problem in their own way. The teacher would ask students questions to find and develop the concept they were learning.

The focus of mathematics teaching in this classroom would be on the students' conceptual understanding.

In Sue's classroom, students started the lesson by interactively exploring a real world math problem and trying to find solutions by creating their own procedures or algorithms while discussing their ideas in groups with other students. The groups would then share their conjectures and how they attempted to solve the problem with the class. Sue then made connections between students' work, using mathematical representations, to help them analyze and evaluate their logic to substantiate proofs while making connections to the mathematical procedure, formula, or algorithm to solve the problem. Then the problem was extended and additional exploration and externalization of students thinking occurred as they deepened their understanding of the concept and procedure. Then independent practice of problems was worked in the format of homework. The problems were then examined and feedback was given on the conceptual and procedural understandings of students' performance. The focus in Sue Johnson's teaching was students' mathematical understandings of concepts *and* procedures. Sue's teaching included the other strands of mathematical proficiency denoted by the National Research Council but I did not specifically elaborate on those connections.

Sue Johnson taught mathematics with a complex process of integrated conceptual and procedural understandings. An important aspect of mathematics for teaching and learning may be the relationship between procedural knowledge and conceptual knowledge because procedures should be developed from, and built upon, students' understanding of the underlying concepts (Ambrose, Clement, Philipp, & Chauvot, 2004; Hiebert, 1999; Hill & Ball, 2004; Lloyd & Wilson, 1998; Rittle-Johnson & Koedinger, 2002). In Sue's classroom, mathematical understanding was a *combination* of developing conceptual and procedural skills and knowledge. One was not separated from the other. At this point in the study, I was curious about this relationship in her teaching so I decided to create a categorical aggregation chart (Stake, 1995) to represent these relationships symbolically. I developed two charts of Sue's first two lessons to find relationships and patterns between conceptual and procedural interactions during the lessons. Only chart one (Table 6; also see Appendix G for detailed chart) is included here because chart 2 showed the same relational patterns. The charts were developed using the audio tapes for observations one and two of her teaching. Each box of the chart represented one minute of teaching. I recorded whether the interaction was conceptual "C" or procedural "P" or a combination "C/P or P/C" of the two during each minute of her lesson. For instance, C/P symbolized that emphasis went from a concept to a procedure. Sometimes it was hard to distinguish if an instance was conceptual or procedural but it was obvious that both were frequently occurring simultaneously. I found that a representation "R" became a significant factor of classroom interaction in her mathematics teaching.

It became clear that Sue Johnson's mathematics teaching used interrelated connections between conceptual understanding, procedural execution and semantics, and interactive representations using mathematical language.

Figure 1 is a representation of how Sue weaved a web of mathematical understanding for her students. Sue interacted with the students and a mathematical problem and students interacted with each other, the teacher, and the problem as represented by the triangle. She enhanced cognitive development of students through

Table 6

Sue's Categorical Aggregation

	1-Minute Intervals				
Minutes 1-5	C/P to P/C	C	C/P	C	C/P
Minutes 6-10	C	P/R	R/C/P	C/P/R	P/C/P/R
Minutes 11-15	C/P/R/C	R/C/P	R/P/C	C/C	R/C/P
Minutes 16-20	R/P/C	C/P/C/R	P/C/P	C/C	C/P
Minutes 21-25	R/P/C	C/C/P/R	R/C	C/C/P	R/C/C/C
Minutes 26-30	R/C/P	C	R/P	R/P/C	C/P/C/C
Minutes 31-35	C/C	R/C/P	C/R/C/P	R/P/C	P/C/C
Minutes 36-40	P/R/C	C/P/R/R	P/C	C/P/C	R/P/C
Minutes 41-45	C/R/P	C	P/C/R	P	C
Minutes 46-50	C/C/R	C/C/P	C/P	P/C	C/P
Minutes 51-55	C/P/R	C/P/C/C	R/C/R/P	C/C/R/C	C/R/C/P
Minutes 56-60	R/C/P	C/P	P/R/C	R/C/P	C/C/P
Minutes 61-65	R/C/C	X	R/P/C	C/P/R	P/C

The Mathematical Focus of Instruction:

C = Conceptual
P = Procedural
C/P = Conceptual to Procedural
R = Interaction with a Representation

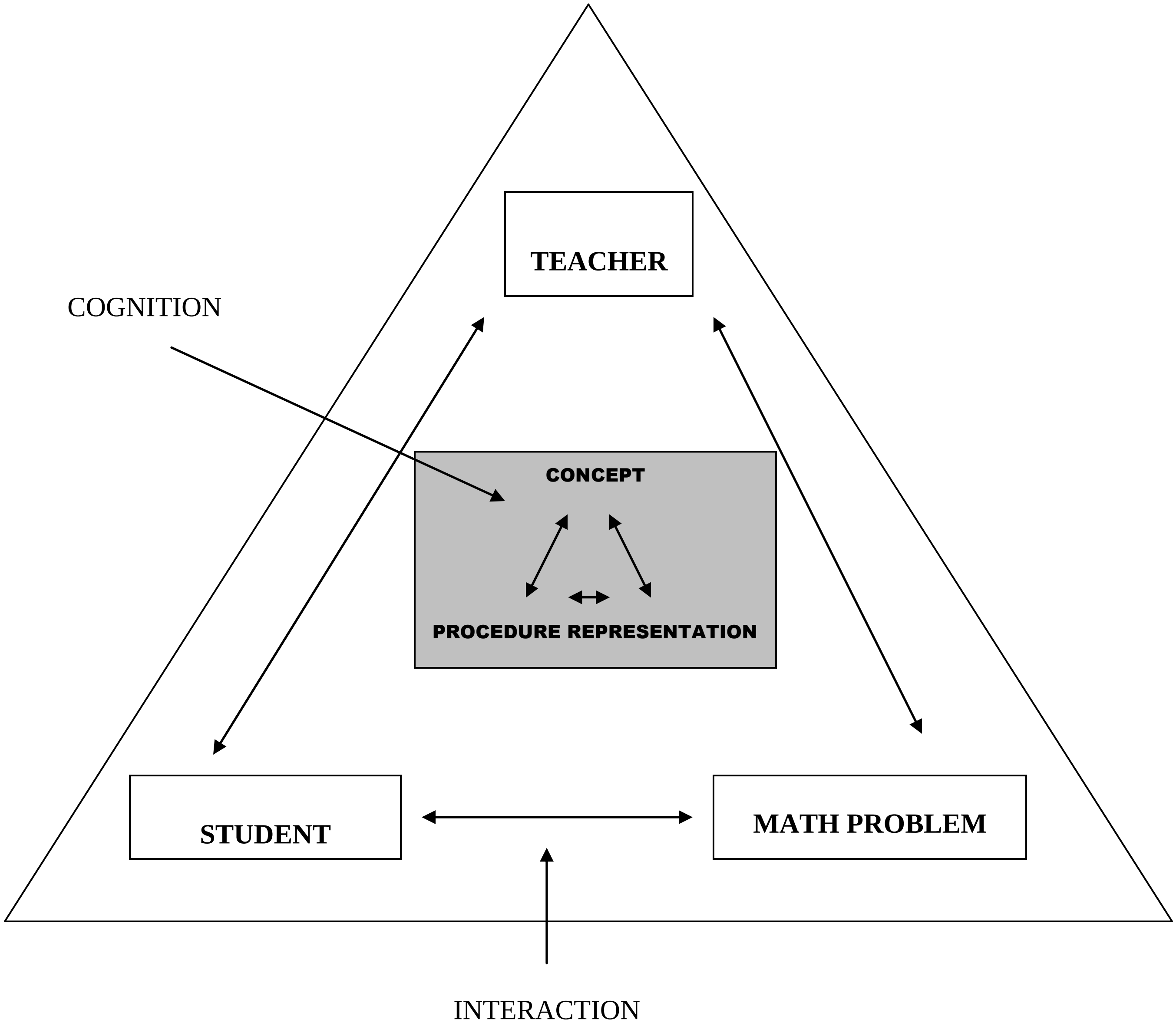

Figure 1. Sue's cognitive teaching model

interdependent and interwoven relationships between concepts, procedures, and representations within those classroom interactions. Each component played an important role of a balanced approach to deep mathematical understanding and helped students make complex connections to all of the National Research Council's strands of mathematical proficiency providing a deep understanding of the conceptual and procedural knowledge and skill of the discipline.

I wrote about each component separately but it is important to note that they were taught naturally together as if they were inseparable in her teaching. Sue wanted students to learn why something was happening, how it is related or connected to other variables, when to use it, why students need to know it, if it was logical, and how to transfer it to another situation during the conceptual phase of her teaching. By having her students create their own conjectures she created a zone of proximal development where she took the students from where their understanding was to where she wanted their mathematical understanding to go. Sue taught math in the context of the students' world following their thinking which externalized students' misconceptions and misunderstandings. She then used students' thinking, examples and conjectures to make connections between the concepts and procedures she wanted them to learn mathematically using representations of their thoughts and the procedural language of mathematics. Student thinking was valued, recorded, and represented in mathematical language for the class to analyze, synthesize, and evaluate allowing her students the opportunity to connect their conceptual knowledge to mathematical procedures (NCTM, 1991). This taught her students that mathematics was about thinking and the relationships between concepts and procedures.

As students verbally expressed their thinking, Sue provided a visual representation of that thinking in mathematical form on the board or overhead for the whole class to see. This allowed students to see how conceptual words are converted to procedural mathematical language. Symbolic representation has been considered especially beneficial in mathematics education since these representations are supposed to allow a concrete-metaphorical approach to abstract principles (Bills, Ainley, & Wilson, 2003; Bonotto, 2003; Da Rocha Falcao, 1995; Gravemeijer, 1994; Selva, 2003). Sue's representations helped students see the problem in new and mathematical ways and became an active way to interact with the mathematics of the problem. This improved students' conceptual and procedural knowledge and explored mathematical rules and procedures in a practical context. Sue would draw a representation of students' conceptions using multiple colors of markers to show what was occurring and in what order to show the students the mental steps to go through to solve the problem. She asked questions of the students throughout this process to guide their thinking both conceptually and procedurally. In J. R. Anderson's view (1993),

> knowledge needs to be proceduralized, especially solving problems in mathematics. Once a student knows the steps to solving a problem (knowledge is proceduralized) and understands when and where it can be used (conditional knowledge), it can be applied rapidly and reliably across a variety of situations.

Sue demanded that students show their work, write on individual white boards, work with manipulatives and draw representations using the appropriate vocabulary to explore and provide proof and logic to their thinking and conjectures. Wu (1996) states, "proof is the backbone of mathematics." This process allowed Sue to teach students *mathematical* understanding. Although a universal definition of mathematical understanding has evaded the discipline for many decades (Meel, 2003), I will define mathematical understanding in this study as understanding of collective connections of mathematical procedures, language, concepts, and symbols. Mathematical understanding is the compilation of all the specialized kinds of understandings collectively in mathematical learning. For example, in the Pepsi problem, Sue jumped at the opportunity to show the students why there was a three cent difference in the two ways that students used to solve the problem and authentically taught that mathematics was a precise discipline with specific rules and procedures. Lampert (1990) suggested that teachers should help their students acquire technical knowledge and skill in mathematics and by acquiring these tools (language and symbols), the individual is able to articulate the meaning of his or her ideas and construct more sophisticated understandings.

Sue used representations to visually guide students' thinking about the conceptual and procedural understandings of the mathematics that she was teaching. The Common Core State Standards for mathematics (2010) suggested that students should reason abstractly and quantitatively. They want students to have the "ability to deconceptualize – to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own." Sue used representations in this way to further students' understanding and made connections between the conceptual, procedural, and other mathematical understandings they were exploring. Sue used representations to externalize the mathematics so the class could examine it in a visual way presenting new ways to look at and think about the mathematics in the problem. Hill and Ball (2004) suggests that to support flexible understanding, teachers need to (a) probe stages of student understanding, (b) comprehend multiple student solutions and methods, and (c) provide powerful classroom models with which to work. For instance, in the proportions problem Sue drew a representation as the students depicted their understanding of the problem. The students came up with the correct answer but by analyses of their thinking and interaction with the representation it was discovered that the students did not understand the conceptual or procedural parts of the problem correctly and were able to experience improved mathematical thinking through cognitive conflict. Students learned that they misunderstood how to set up the problem correctly and how to modify the representation when one of the variables had changed.

Sue was able to show students mathematically through representation when students had multiple ways to solve problems which developed multiple connections for the students in the schematic structuring in their brains. Sue's teaching of mathematics was about more than mathematics, it was about students developing their thinking by making choices, creating conjectures, interacting with and solving problems, and metacognition while building mathematical understanding. Senge et al. (2000) wrote, "that if you want to teach people a new way of thinking, don't bother trying to lecture or instruct them. Instead, give them a tool, the use of which will lead to new ways of thinking" (p. 331). Sue used representation in her mathematics teaching that became a tool for her students to think differently about the problem. Sue wanted her students to know more about mathematics than to find the correct answer. She wanted students to explore and discuss the underlying mathematics through a problem solving format to achieve a deeper understanding of mathematics. Hung-Hsi Wu (1996) believed that the most serious issue facing a problem-oriented curriculum was that the problems are only a means to an end. Wu states, "Therefore the solutions to problems in such a curriculum need to be rounded off with a mathematical discussion of the underlying mathematics" (p. 9). Sue Johnson's teaching is one example of using mathematical discussion to identify and develop underlying mathematical ideas.

Dimensions of Excellent Elementary Mathematics Teaching

Dimensions of excellent elementary mathematics teaching in this case were developed through direct observational interpretations, specifics of each teacher's classroom experiences and key issues, and reflection of the cross-case analyses. Key issues and observational data for each individual teacher were examined collectively in a cross-case analysis forming a set of categorical correspondence or patterns. The categories created by the cross-case analysis were combined with interpretation of Sue Johnson's unique blend of intertwined teaching and compared with the research literature to form six teaching dimensions. These eight dimensions are a compilation of excellent elementary mathematics teaching from this case study. They were created by progressively focused analyses of integrating the categories, direct interpretations, and by comparing the findings with research literature on mathematics teaching.

Dimension 1: The teacher is the curriculum. The teachers in this case taught with little reliance on a textbook, spontaneously taught to students' needs, identified and presented multiple ways to solve problems, and made immediate adjustments to instruction or activities as they were teaching. They were the curriculum because they taught mathematics in ways that were interactive and alive with continuous curricular and pedagogical decisions being made by the teachers during the teaching of mathematics. They had mastered deeply the mathematics that their school district required. It was how they taught the mathematics that made each of these teachers the curriculum in their own classroom.

The teachers were the curriculum because of their ability to use adaptive instruction to the written and intended curriculum of their school district. They had at their disposal many options of how to teach students the mathematics. They did not know which options they would use until they were actually teaching the lesson. Kathy stated, "You never quite know how you're going to get there. Certainly, you have to have some ideas but you never really know until you're in the middle of it." To understand why these teachers have chosen to be the curriculum in their mathematics teaching we need to examine their teaching beliefs and philosophies. Teachers' beliefs and values may also be an essential aspect of their classroom practices (Ambrose et al., 2004; Ross, McDougall, Hogaboam-Gray, & LeSage, 2003; Stipek, Givvin, Salmon, & MacGyvers, 2001). They believed that mathematics should be taught for understanding of the concepts and procedures with emphasis on the processes and wanted students to experience mathematics differently than their experiences. This allowed students to think and make connections to mathematical ideas so that they would be able to transfer the skills to their lives and other mathematics. The teachers believed that mathematics was not a stagnant discipline and that new strategies, thinking, and concepts were being developed continuously. Mathematics involved precision in language, thinking and execution for them. The teachers valued each of their student's ability to think and believed they all could succeed in mathematics. Mathematics learning in these classrooms involved social communication where students and their ideas were valued and explored whether they were correct or not. Their belief that mathematics should be taught and learned experientially where students are actively engaged and the teacher is a facilitator and guide was evident.

All three teachers taught mathematics traditionally from a textbook and followed a teacher's guide when they started their teaching careers. Through their mathematics experiences, teaching experiences, training, research, and their developed philosophies of teaching mathematics they developed the confidence to take the risk of teaching mathematics differently. Cohen and Ball (1990) stated, "How can teachers teach a mathematics that they never learned, in ways that they never experienced?" These teachers' deep passion for the discipline of mathematics and their students allowed them to seek and find ways to learn and teach mathematics they had never experienced before believing there was a better way to teach and learn mathematics.

The teachers could be the curriculum because of their deep understanding of the multidimensional knowledge of the mathematics they were teaching. The Report of the National Mathematics Advisory Panel (2008) states that "Teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching." They were able to spontaneously change course during a lesson easily because of their multidimensional understanding of their curriculum. These spontaneous changes or diversions occurred in activities, assessments, assignments, and communications during the teachers' teaching. Sue said, "I know what I want to teach that day. It kind of depends on how students respond and sometimes I change my plan in mid-course." "I don't really plan my questioning ahead of time. It's nothing I can read in my lesson plans in my teacher's guide that tells me how to ask questions. They just happen," said Kathy. Callie said, "I have my plans down but it doesn't mean they're going to be followed, exactly."

The teachers chose to teach mathematics through exploration and discovery instead of using a textbook format because of their deep understanding of the curriculum. Sue replied when I asked about using a textbook, "They call it scripted which tells you what you can do and what you should do next and I don't do that. I have a better understanding of math and I feel comfortable that I don't have to use a scripted book." This type of teaching requires lengthy and detailed planning of tasks, questions, materials, and thinking about how students may react to the lesson. The National Research Council (2001) states that, “Teachers who teach this way must prepare in detail for class; many observers of teaching fail to appreciate the significance of design and preparation in making these sorts of lessons more effective in helping students learn” (p. 424). Callie said, “On the weekends I sit down and spend hours in depth planning through each lesson writing out what I’m going to say, what kids are going to do. I spend a lot of time planning. I feel prepared if I have planned well. I have to know where it is going and the purpose of it.” The teachers needed to be able to make many decisions and adjustments to an open-ended pedagogical format that is interactive and focused on students’ thinking and understanding. The teachers had to sacrifice and work much harder to teach mathematics this way. Callie continued,

> I’m willing to take that risk. To take that jump to go in a new direction with teaching. It’s easy to pick up a book and do the problem, assign problems and grade them, and move on. This isn’t easy. You have to be prepared with questions you’re going to ask, questions you’re going to ask if they don’t get it to guide their thinking. You have to be prepared with how you think they might answer questions or where they might be going with it. If I’m not prepared then I’m not going to do any justice for them. You have to be ready to listen to kids’ math thinking and realize that it might be different than yours. So then you have to be ready to ask those questions to get deeper into their thinking.

The teachers in this study had a number of pedagogical options available to teach each concept and procedure. A lesson usually started with a problem solving activity and was followed by a number of activities that were determined by how students were interacting and thinking about the problem. These teachers chose student behavior to guide the direction of the lesson because they valued individuality and focused on their students’ needs. This strategy advanced the understanding of the mathematics students learned while interacting with the problem as teachers reflected spontaneously to what was happening in the lesson. Their strategies included: data collection, conjectures, finding proof, using students' examples of their work, many variations of questions to specific students, groups or the class, discussions, representations, manipulatives, and showing multiple ways, as well as others. Activities were selected for students to explore and discover the mathematics and to develop their current understandings while in the zone of proximal development.

During "teachable moments" the teachers jumped at the opportunity to show how the problem was being solved with non-routine algorithms, students' mathematical thinking, and identified additional pathways to solving math problems. Silver et al., (2005) claims that, "different solutions can facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 228). Research also indicates that mathematics instruction that values individual knowledge construction, enabling students to find correct mathematical procedures in different ways has been strongly advised for the teaching and learning of mathematics (Schoenfeld, 1983; NCTM, 2000; Leikin, Levav-Waynberg, Gurevich, & Mednikov, 2006).

The teachers were continuously assessing students informally while all of these activities were occurring in the lesson. This allowed them to make immediate decisions about what to do next in the lesson to increase student understanding. "I try to reflect or be observant of how the students are reacting and go from there," said Sue. Kathy noted,

> I'm reflecting every second that I'm teaching. It's happening constantly. If I'm seeing what I'm using isn't working then I have to figure out something that will work. Then I'll start thinking about how I'm going to get them to have some meaning to that objective. I am everywhere in the room so I usually see it when its happening.

The teachers assessed what students were doing and what students were saying about the problems, tasks, or questions they were posing. When Callie was teaching a lesson on equality she noticed by observation that the students were using the rubber bands and geoboards to achieve symmetry that they had done in a previous lesson.

> I was walking around and almost everybody was doing symmetry. So I had to stop them half way through and give them an example of what I meant by using one rubber band to create equal not symmetry and they did well after that. Sometimes they just need to see an example, especially if their mind is thinking another way.

These kinds of assessments were happening continuously throughout each teacher's lessons. In this case, the teacher used an overhead projector to show her students an example of an equality while making distinctions between an equality and symmetry for students.

These assessments led the teacher to adjust the lesson to adapt to how the students were conceiving the mathematical concepts and procedures of the lesson. The National Research Council (2001) suggests that "after interpreting students' work, teachers need to be able to use their interpretations productively in making specific instructional decisions: what questions to ask, tasks to pose, homework to assign." Some of the adjustments that these teachers used were: adjusting the task, calling a time-out, showing multiple ways, asking additional questions, extending the problem, elicit explanation of thinking, writing, and group work before completing formative assessments. Kathy said,

> I start my teaching and that's when I start making my adjustments. I don't really make them that much in the planning. I make them in the teaching as I'm teaching. No lesson goes quite the way I expected it to. It's never cut and dry like the book makes it sound. There's always a stumbling block.

The teachers chose the option that they felt would benefit students' understanding to the specific situation. For instance, when Kathy was talking about her analysis of students' whiteboard work she replied,

> First of all I have to figure out why they wrote that. Why did they get that? Where do I go from here? I know how to solve it when I know where they are coming from. What little piece do they need to have a better understanding of that. Those boards are really important to me. I can see their thinking and then I know what direction I need to take.

The teachers always had the students' best interest in mind as they made their instantaneous decisions. Senge et al. (2000) suggests, "Too many have forgotten that they are teaching students as well as a subject" (p. 93). Students were not only the main reason for each of the teachers' pedagogical decisions but were treated as valuable and respected people. Senge continued, "All learners construct knowledge from an inner scaffolding of their individual and social experiences, emotions, will, aptitudes, beliefs, values, self-awareness, purpose, and more." These teachers nurtured and guarded their students' cognitive, social, emotional, physical, and psychological growth which enabled students to feel valuable, included, and comfortable experiencing complex mathematical study. Palmer (1993) noted, "Good teachers bring students into living communion with the subjects they teach. They also bring students into community with themselves and with each other" (p. xvii). The teachers were teaching more in their curriculum than just mathematics, they were teaching children. They made sacrifices that taught the mathematics curriculum in pedagogical ways that benefitted their students, made assessments and adjustments by what students said and did, and supported and enabled their students to be valuable and protected people. "What you understand is determined by how you understand things, who you are, and what you already know as much as by what is covered, and how and by whom it is delivered," (p. 21) concluded Senge. Sue showed how she favors pedagogy that benefits her students when she talked about how her students preferred non-routine algorithms in solving problems,

> When I teach the lattice method of multiplying to my students I get a lot of parents that say what are you doing and they don't understand this and they think it is a more difficult way to do it, but my students think it is an easier way to do it. It helps them organize things and when I write a problem up there they ask, can we do it in a lattice box, because they don't want to do it in the traditional way. They get lost in the traditional way.

The teachers always had the needs of the students in their thinking as they were teaching. Callie said, "Sometimes it's good to take it (mathematics) where the kids want to take it. I think they feel that their thoughts are validated if they are addressed if they bring them up." The teachers believed that each of their students could learn mathematics successfully. By valuing students' thoughts and making sure that each child was able to participate and share their ideas mathematics became a discipline that everyone regardless of current ability could improve their mathematical understandings.

The National Mathematics Advisory Panel (2008) state, "Experimental studies have demonstrated that changing children's beliefs from a focus on ability to a focus on effort increases their engagement in mathematics learning, which in turn improves mathematics outcomes" (p. xx). These teachers focused on students' efforts to understand what they were doing better and, therefore; students believed they were getting smarter. In one of Callie's lesson a student commented, "We're getting smarter all the time, Mrs. Hendricks!" This contradicts society's belief that success in mathematics comes from inherent talent or ability and allowed the students opportunities to grow mathematically. The pedagogical choices the teachers chose helped students improve their own understanding of the mathematics that was being taught. Kathy said,

> I think you have to be able to fill in that gap because if you can't fill in that gap then you can't help those kids that aren't understanding. There's a gap to fill every single moment of every math lesson. I've got to come at it from a different direction. It might be one direction for one person and another direction for a different person. I have to keep thinking.

It was how the teachers taught mathematics to their students that made them the curriculum. The teacher as the curriculum was more than the academic content for mathematics. The National Research Council (2001) states that,

> The quality of instruction does not inhere in any single element, challenging, exemplary curriculum material; competent, enthusiastic teachers; or capable, eager students. What makes curriculum expemplary, teachers competent, and students capable is their skilled use of one another to produce teaching and learning. (p. 333)

They taught mathematics through students' action, reflecting on assessment, responding by adjustment, and applying what was learned to another action to improve individual mathematical understandings. They were interacting with students and allowed students to interact with each other all with the goal of better understanding the mathematics the teachers were teaching.

Dimension 2: Used cognitive approaches. In this dimension of excellent elementary mathematics teaching I will demonstrate how these teachers used constructivist and cognitive approaches in their mathematics teaching. In my analyses of the teachers' teaching I noticed that their teaching was more than a deliberate set of mathematical activities to engage students in discovering and exploring mathematics. The teachers had a genuine interest in how the students learned the mathematics *and* how students were processing that information. These teachers could easily be identified as constructivist teachers because of their epistemological beliefs about how students learn mathematics, but these teachers also monitored and focused on how students were developing or processing the mathematical concepts and procedures that they were teaching.

If you were to ask 20 teachers to give you a definition of constructivism you would probably get 20 different answers with similar features. This is because constructivism is a difficult concept to grasp. Terms such as these have multidimensional complexity attached to them and are difficult to develop one specific definition that encompasses the complete meaning. Literature identifies this difficulty by offering many different definitions of constructivism. It is important that we distinguish between definitions to capture the complexity of these teachers' excellent mathematics teaching. Constructivism has been influenced by the work of Dewey, Piaget, and Vygotsky, among others, has become the leading theory on learning and has evolved during the last few decades. Teachers once thought they could be constructivist teachers by applying "hands-on" activities where students could construct their own understanding of what was being taught. Later the term became "hands-on, minds-on." Definitions that elicited this kind of conception of constructivism focused on what teachers and students do in a constructivist environment. Definitions used terms like; "active, hands-on, construct their own knowledge, facilitator, guide" to define constructivism. It was assumed by many teachers that this meant that constructivism was a teaching theory that was represented by a set of activities that the teacher presented to the students differently than traditional pedagogy. The teachers in this study did use these kinds of pedagogical strategies but is an incomplete picture of how these teachers taught mathematics.

The definition and conception of constructivism has evolved in the last decade to a theory of "learning." Fosnot (2005) states, "Constructivism is a theory about learning, not a description of teaching. No "cookbook teaching style" or pat set of instructional techniques can be abstracted from the theory and proposed as a constructivist approach to teaching" (p. 33). Constructivism is more complex than a set of pedagogical strategies. Some of the current definitions of constructivism include words that describe cognitive processes or cognition. Brooks and Brooks (1993) define constructivism as a theory of learning that describes the central role that learners' mental schemes play in their cognitive growth. The Association for Supervision and Curriculum Development's The language of learning defines it as an approach to teaching based on research about how people learn. The definition goes on to describe active, hands-on learning during which students are encouraged to think and explain their reasoning. These definitions indicate that constructivism and cognitive approaches to teaching have similar traits and are converging together in a new conception of what constructivism is. Cognitive psychology indicates a series of themes for education (Bruning et al., 2004),

1- Learning is a constructive, not receptive process
2- Mental frameworks organizes memory and guide thought
3- Extended practice is needed to develop cognitive skills
4- Development of self awareness and self regulations is critical to cognitive growth
5- Motivation and beliefs are integral to cognition
6- Social interaction is fundamental to cognitive development
7- Knowledge, strategies and expertise is contextual

Brooks and Brooks (1993) identified five central tenets of constructivism,

1- Seek and value students' point of view
2- Structure lessons to challenge students' suppositions
3- Recognize that students must attach relevance to the curriculum
4- Structure lessons around big ideas, not small bits of information
5- Assess student learning in the context of daily classroom investigations

Notice that the constructivism tenets stress students and lessons while the cognitive principles focus on learning and cognitive development. The excellent teachers in this study focused their teaching and learning on both sets of principles. The new approach to excellent teaching includes cognitive approaches to learning using constructivist pedagogical strategies.

These teachers used cognitive approaches to their mathematics teaching in a constructivist way. I will demonstrate in my writing how these teachers used constructivism and cognitive approaches in their teaching separately to reveal a complete picture of their teaching. It is important to note that these approaches were functioning simultaneously and were not separate in their teaching. I will use the two sets of principles above as a model to describe how these teachers taught mathematics.

The teachers wanted to know and valued each student's point of view. This is why they used open-ended problems and had students interact with the problem without telling students what the mathematical algorithm was. They were externalizing students' thoughts, preconceptions and prior knowledge about the mathematical concept or procedure. They did this with questioning techniques, individual boards, listening to students talking in different configurations of groups, and having students show their work in many forms including manipulatives and representations. This allowed the teachers to attend to the specific needs of their students and progress their construction of knowledge while deepening their understanding. It also helped the teachers guide students' thinking on what was important while facilitating the students' attention and perceptions.

The teachers' structured lessons to challenge and expose students' suppositions. Using problem solving problems in their teaching brought out the suppositions that students had about the mathematics. For instance, in the 7 divided by 4 problem students' suppositions about dividing a whole into fractional pieces were exposed and developed. They learned that to solve the problem correctly you had to start with 7 wholes and that by dividing the whole into four parts to accommodate the four people made sense. They couldn't just throw away the leftover pieces or add other pieces to solve the problem. The teachers were able to access students' suppositions because they asked questions and created problems that the students interacted with without telling students how to do it. McCann, Besner, and Davelaar (1988) state, "Knowing what we see (or hear) and even how to look (or listen) depends on the knowledge we have." The teachers wanted to know students suppositions, prior knowledge, and their point of view because that allowed the teachers to work with the students in the zone of proximal development with the mathematics they were teaching. Spontaneously choosing tasks that built on students' prior knowledge helped their students engage in mathematical tasks for longer periods at higher levels of thinking.

The teachers wanted their students to reason mathematically. They created disequilibrium through questioning and tasks for students that required them to reason and defend their conceptions. Factors that motivate students to use higher order thinking and mathematical reasoning include classroom environment, teacher questioning that evokes meaningful support of conjectures, and well designed tasks (Mueller, Yanelewitz, and Maher, 2011). These excellent teachers wanted their students to experience and reason mathematically instead of having them rely on the teacher. Through participating in classroom activities, mathematically autonomous students begin to rely on their own reasoning rather than on that of the teacher (Forman, 2003) and thus become arbitrators of what makes sense. Other research shows that when students learn the reasoning and proving in mathematics, they will be proficient in mathematics (Kamarnddin, Kamariah, & Zulkarnain, 2012; McCosker & Diezmann, 2009). The students in these teachers' classrooms relied on their own reasoning instead of memorizing facts to convince themselves and others of what made sense. This type of reasoning leads to mathematical understanding (Mueller, Yankelewitz, & Maher, 2011). The teachers in this study expected their students to reason mathematically as they focused on how students were cognitively constructing the mathematics being taught.

The teachers emphasized mathematical tasks that were relevant to students' interests and worlds. Sue created problems about farming because that was where most of the students lived. Other problems were centered around candy bars, pop, sharing, friends, sports, and other games and activities that are a part of children's everyday lives. Mathematical ideas were presented using the students, their work and thinking, and items in the room solving everyday real world problems. By using students' prior knowledge and progressing students' thinking through suppositions and conjectures also made the mathematics relevant to the students. Brooks and Brooks (1999) state, "Students must be permitted the freedom to think, ask questions, to reflect, and to interact with ideas, objects and themselves—in other words, to construct meaning" (p. 103). The students in these teachers' classrooms were allowed to do all these things which gave them additional mathematical relevancy. Brooks and Brooks added, "Initial relevance and interest are largely a function of the learner's experience, not of the teacher's planning." This is why the teachers chose to teach spontaneously and divert regularly.

The teachers used problems as the big mathematical ideas instead of using a linear approach to instruction. Within the structure of the whole problem the teachers exposed the students to see the mathematical parts or steps needed to solve the problem. Von Glasersfeld (1995) believed that constructivism cannot be shared and is about "cognitive development, deep understanding, constructions of active learner reorganization, complex, and nonlinear." Using big ideas helped these teachers' students reorganize their understandings more deeply in complex and connected ways. For instance, in the Pepsi problem the teacher asked students to discover the "better buy" of the two pop options. Students had to break down the problem using mathematical concepts and procedures of proportions and ratios to determine which product was the better buy. Sue changed the prices and measurements, extended the problem, showed multiple ways to solve the problem, demonstrated why there was a three cent difference between two different ways to use the mathematical skills, and used representations to show symbolically what was happening mathematically. These events were generally nonlinear and made connections between concepts, procedures, and problem solving mathematically. Fosnot (2005) states, "Learning – deep, conceptual learning – is about structural shifts in cognition. It is about self-organization at moments of criticality. Meaning is understood to be the result of humans setting up relationships, reflection on their actions, and modeling and constructing explanations" (p. 279). The teachers broke down mathematics into parts and used the concepts and procedures mechanically to solve problems that changed and developed individual structures of understanding. Kathy said, "I'm able to break down the learning into pieces to something they can understand. I can look at what they are doing and know what they thought to get it and how to help them from that point."

The teachers assessed students' learning individually and collectively in the context of daily classroom investigations and interactions. Students were able to demonstrate their understandings of the mathematics continuously in many different ways. The contextual and authentic assessments included but were not limited to: discussions, explanations, representations, questioning, writing, dialogue, daily assignments, and projects. Brooks and Brooks (1999) state, "Defining understanding as only that which is capable of being measured by paper-and-pencil assessments administered under strict security perpetuates false and counterproductive myths about academia, intelligence, creativity, accountability, and knowledge" (p. 103). The focus of these teachers mathematical teaching was on developing students' individual understandings of the mathematics they were teaching, to guide their teaching, and improve student learning in an authentic holistic way. Their teaching and learning of mathematics deemphasized scores and was about students developing their understandings in these classrooms. All three teachers told me that most of their students were proficient in achievement assessments and they wanted their students to perform and be successful. Their instructional decisions did not emphasize performing on specific achievement instruments except for Kathy's teaching where students completed daily worksheets in the first grade to familiarize themselves to the format of the assessments they would encounter. They used students' work, conjectures, suppositions, points of view, solutions, and discourse in addition to some formal assessments to assess what they know and did not know and to guide instruction.

These teachers also taught mathematics with a focus on their students' cognitive development and processing of conceptual and procedural mathematical understandings both individually and collectively. Kamii (1989) suggested that "if we encourage [students] to develop their own ways of thinking rather than requiring them to memorize rules that do not make sense to them, children develop a better cognitive foundation" (p. 14). The teachers wanted students to develop multiple ways of knowing and cared about how students were developing their conceptual web of understanding schematically. Kathy said, "Parents are surprised at what level of understanding a first grader can have. Some say that their child is bored because they think their child already knows it and at some level they do but at other levels they don't." Attention to detail and precision of how the students' brains were processing the mathematics was of major importance to these teachers. Sue said,

> That one year I tried to use all three methods with my 4th graders but what happened was they were getting all the ideas mixed up and then couldn't remember which one to do. I wanted them to know the traditional method and the lattice method well. I just don't think they're [4th graders] ready to have that many different things in front of them. Conceptually their brains aren't ready to handle that many different methods.

The teachers worked diligently to develop the mathematical schemata in each individual student. They wanted students to develop set schema to represent the idea of part to whole relationships. When using addition you have the presentation of two or more sets or parts that are combined to form a whole (superset). Kathy's use of a Part-Part-Whole (PPW) mat externalized this process for her first graders. Her student who added another part to make the mat a PPPW mat would be classified in cognitive psychology as change schema showing different ways that parts may be combined. These two schemata have been described as encompassing much of arithmetic, and potentially much of other mathematics operations (Bruning et al., 2004). They propose that, "In general, we argue that mathematics operations are not rote learning but rather require the acquisition of networks of mental representations." I used Bruning et al.'s (2004) cognitive themes as I demonstrate how the teachers used cognitive approaches to their teaching of mathematics. You will notice that some of the cognitive themes are similar to Brooks and Brooks (1999) constructivism's tenets.

The teachers believed that learning mathematics is a constructive, not a receptive, process. They exposed what students currently understood about a problem as they explored and conjectured ideas and uses of mathematical skills to construct meaning to concepts and procedures. They despised the traditional process of teaching mathematics and favored a model of constructing understanding instead of teaching a series of isolates procedural skills. Kathy talked about how she disliked having a "prescribed" curriculum,

> My teaching lesson scheduled for me today is lesson 19.6. I've been given an introduction, how I'm suppose to give the introduction, what what they're suppose to do and how to end it. It's pretty much prescribed. However, the activity I did today was not prescribed. I didn't sense they truly understood how and when to use the strategy. How I go about teaching what is prescribed for me I have some leeway there.

I did not see Kathy teach a prescribed lesson in her classroom. I believe that she did not feel comfortable about expressing her diversion of how she taught the mathematics openly. These teachers wanted students to experience the concepts on their own and present their own conjectures and algorithms for problems and used those discoveries as the focus of the continuance of the lesson. Bruning et al. (2004) suggests that knowledge is created and re-created on the basis of previous learning, not simply acquired. What motivates learning is the search for meaning. These teachers focus at all times was on the students' development of the meaning of the mathematics that was being taught. You notice in chapter three that virtually every lesson, question, problem, representation, discussion, etc. was about meaningful understanding of mathematical concepts and procedures. For instance, in the 4% problem, Sue wanted students to understand the meaning of the calculator's answer of .4. What did it mean? Is it related to other representations of the same value? How do we apply it? Is it a reasonable answer? It was not all about the answer and more about what it means, how to apply it, and when to use it. The teachers were teaching students that mathematics and learning is about thinking and students learned how they could apply learning to what they already know, organize it, and check their comprehension of it. This is what the teachers meant when they said that their teaching held kids accountable for their learning. They taught mathematics as facilitators and guides and rarely presented answers to problems. They helped their students find their way through problems without telling them how to do it.

The teachers focused on how students were conceptually processing the mathematics they were teaching. They were assessing how students were organizing their thoughts and information that was being processed. That is why they continuously had to know what the students were thinking all the time. The teachers externalized students' thinking to assess their understandings through verbal and visual prompts and expected students develop mathematical reasoning. How students were processing the lesson determined how they would proceed to develop precise conceptions into students' mental schemata. It is believed in cognitive psychology that schemata are mental frameworks we use to organize knowledge (Bruning et al., 2004). It was important to the teachers that students were building precise conceptions of mathematical knowledge. Wu (1996) states, "Precision is a defining characteristic of our discipline." He went on to separate precision into two categories: conceptual precision (definitions, theorems, and proofs) and formal precision (symbolic computations and algorithms). These teachers emphasized both types of precision to their students. For example, Sue jumped at the opportunity to show the mathematical reasoning and symbolization for the three cent difference in the two methods in the Pepsi problem stressing that for some problems you may have to precisely calculate problems to the hundredth place value if small variations of an answer are important. In the probability problem representation, the student changed one variable and the teacher explored with her students how that change affected other symbolic variables. In these two teaching episodes, as well as many others, both categories of precision were implemented in the teachers' teaching and students' learning of mathematics. These teachers taught mathematics with the focus on the students instead of on the curriculum. Researchers today have shifted their attention from learning to focusing on "learners themselves—to their prior knowledge and frames of reference, to the activities they undertook and the strategies they used as they learned, and to their role in creating new knowledge," states Bruning et al. (p. 6). These teachers were always focused on their students and their specific needs and understandings.

These teachers used extended practice of the mathematical skills sparingly. Bruning et al. (2004) determined that extended practice is equally important for the development of cognitive skills;

> Although we typically think of cognitive psychology's emphasis on meaning and thought, the other side of cognition-automated processes-is equally important. Automated processes in attention, perception, memory, and problem solving allow us to perform complex cognitive tasks smoothly, quickly, and without undue attention to details. (p. 25)

The automaticity of specific skills and concepts allows a person to focus their attention and cognitive energy to more complex mathematical thought. Sue talked about computational skills and teaching her students in the different grades,

> I tell my fourth graders that we'll work on computation up to fourth grade and then I expect you to know your computation. Because if you can't add, subtract, multiply, and divide then everything you do in 5th and 6th grade and every grade beyond that is going to be difficult.

Sue felt that as students develop mathematically through the grades that the focus of the mathematics is more on processes and "not computing what five plus seven is." Sue talked about basic computation and mathematical processes, "Take area and volume, there are some steps you have to take (processes) and computation kind of gets in the way, but it's important. If you don't have that part under control it makes process difficult." Sue indicated how important the computational skill is to the students' future mathematical success so students could focus their mental capacities on more complex mathematical thought and processes. This is why she allowed students to use calculators at specific times in her mathematics class because she wanted cognition to be on mathematical conceptions.

Extended practice in these classrooms were in the form of extended problems or solving another problem of the same concept or process in a different context. In Sue's classroom, students had daily homework where extended practice was used to enhance automaticity. Kathy had a daily worksheet of about 10 problems depicting the skill of the day and timed tests of basic facts to develop these skills for first graders. Callie did not use any additional time to develop computational skills. The teachers assigned a small number of computational practice problems and focused student skill on transferability. They wanted students to understand the mathematics deeply so that they could transfer the skill to other contexts and real world situations. The teachers felt that their students would obtain the extended practice of repeating sets of mathematical problems in the paper and pencil format in the other grades of their schooling and that it was more important to focus on the conceptual processes of the discipline while the students were in their class. Callie said,

> I was not worried at all about memorizing facts going into 4th grade. They can work on their facts at home, at night, but in class we don't work on them. We work on understanding, just having that underlying knowledge and applying it to other situations if they have that true base of understanding versus memorizing facts. I think that will help them a lot with problem solving.

All of the teachers told me that computational skill was an important part of learning mathematics and some time was devoted to extended practice at varying degrees in their classrooms.

These teachers helped develop self-awareness and self-regulation in students' mathematical cognitive growth. They created a classroom environment where students were self-directed, creative, strategic and reflective. They wanted students to develop ways to solve problems and experience mathematics on their own, to develop their own strategies and evaluate their effectiveness. Metacognition is an individual's knowledge about how he or she learns and thinks. Bruning et al. (2004) refers to metacognition as two dimensions of thinking: (a) the knowledge students have about their own thinking, and (b) their ability to use this awareness to regulate their own cognitive processes. The teachers' choices of pedagogical activities allowed self regulation and awareness to develop in their students. They expected their students to think and discover mathematics as a community of learners. The teachers facilitated students' thoughts, plans, and conjectures and share them with the class to make connections to the mathematics and further everyone's understandings. The teachers also created cognitive conflict requiring students to analyze their own conceptions and strategies. They posed questions to activate possibilities to prior knowledge, mathematics, and to students' cognitive processes and helped students organize this information into mental frameworks for deeper understanding. These teachers taught more than mathematical knowledge and skill acquisition. They taught their students learning strategies, the ability to reflect on their decisions and others, and to think critically. “Unless learners monitor and direct their cognitive processes, they are unlikely to be either effective learners or flexible, effective problem solvers” (e.g., Boekaerts, Pintrich, & Zeidner, 2000). The teachers in this study wanted their students to know mathematics, learn how to think, and to develop multiple ways of knowing.

These teachers motivated their students to want to learn mathematics and believed they were capable of learning important mathematics they will need as they grow and move into the real world. The focus of their mathematics teaching was on developing individual understandings of concepts and procedures. Student success was not solely measured by how many correct answers a student was able to produce at a given time. Students were able to perform and show their mathematical progress through how they were developing their understandings through multiple authentic informal assessments. The focus was on improvement of understandings of mathematical concepts and procedures. This allowed students to explore and discover mathematics without undue worry about missing the correct answer to problems enhancing their motivation to learn. Research in cognitive psychology has shifted to include student motivation and beliefs that shows students constantly judge their performance and relate them to beliefs about their potential (Bandura & Wood, 1989; Dweck, 2000; Pintrich & Schunk, 2002; Zimmerman, 2000). These judgments are an integral part of whether activities are attempted, completed, and repeated (Bruning et al., 2004). The teachers built confidence in their students that they could learn deep mathematical knowledge and skill and guided them in the development of their cognitive schemata. Students' thoughts and work were valued and analyzed by the class to develop mental frameworks and connections. Students learned how to comprehend mathematical content and also how to become active, motivated, self-regulated, and reflective learners that are valuable thinkers.

One of the major components of these teachers' teaching was social interaction. Rarely were students sitting at their desk working individually on an assignment. Group work, discussions, explanations, and questions that elicited thinking and clarification were an integral part of these classrooms. Cognitive research has shown that these social-cognitive activities stimulate learners to clarify, elaborate, reorganize, and reconceptualize information (e.g., Calfee et al., 1994; Cowie & van der Aalsvoort, 2000). This peer interaction allowed for active social participation in knowledge acquisition and helped students learn new ideas and thinking from each other. The students in these classrooms had opportunities to express their own ideas, listen and observe other students ideas, and receive feedback to develop meaning. Callie said, "Kids can learn from each other. Sometimes they can understand another student's explanation better than mine." These teachers allowed students to share their thinking with each other on their own as they carefully observed and listened to what they were saying and doing to monitor their progress. Frequently, the teachers would stop the social activity and pose questions or highlight student work for unique thinking or algorithms and made sure the whole class had the opportunity to experience a particular group's discovery. They also would stop activities if students' thinking needed adjusted or developed to achieve the mathematical conceptual and procedural objectives.

At times the lesson highlighted other important mathematics that was not planned. These “teachable moments” showed the teachers’ nonlinear approach to teaching and learning mathematics. For instance, Sue was able to indicate mathematical precision as an important component to mathematical calculation for specific circumstances in the Pepsi problem. These nonlinear mathematical experiences occurred in all the classrooms at opportune times when social interaction was occurring. Sometimes they occurred from a student’s response or how a group was thinking about a problem. Questions from the teachers to the students would also reveal teachable moments regarding how students were conceiving or processing the mathematics. Another example was when Kathy showed the class a part-part-whole mat where a student added an extra part to the mat (PPPW mat). She was able to show the student’s discovery and related it to how we add more than two numbers together all the time proving her reasoning to be logical in applying the strategy to other situations. These teachers’ classrooms were places that nurtured a supportive environment for students to discover and explore mathematics safely without ridicule. This was achieved by valuing and developing each student’s mathematical thinking.

The teachers in this study contextualized the mathematical knowledge, strategies and expertise in their teaching. Kathy’s previous PPW mat episode is an example of teaching and learning math contextually. A student adapted an existing strategy to solve a different problem. Her understanding of the strategy allowed her to use that strategy effectively and creatively in a different context. Research has showed that memory was strongly influenced by the actions of learners as they attempted to encode information (Hyde & Jenkins, 1969; Tulving & Thompson, 1975). Learning and memory are enhanced by what learners construct in a social context from their prior knowledge, intentions, and the strategies they use (e.g., Gauvain, 2001). This conception of cognitive development adds to the metaphoric comparison that the brain functions like a computer. It was believed that the brain is an information processor that processes, stores, and can retrieve information. Contextualism emphasizes that there is more to how the brain functions than input, storage, and output. The context of the learning situation affects what is processed, stored and retrieved. The context in which learning takes place becomes some or even much of the meaning of what is learned (Gauvain, 2001; Rogoff, Bartlett, & Turkanis, 2001). The teachers taught their lessons contextually. They wanted students to understand the mathematics so they could transfer the concepts and procedures to different contexts. This was a part of their planning and teaching of the curriculum. For instance, Callie used a series of problem activities so students could use the concepts and procedures in different contexts as she developed deeper understandings of fractions. This introductory unit allowed her students to explore the concepts and numerical relationships of fractions in a number of contextual situations. These lessons were scaffolded and developed concepts and procedures that were learned in previous lessons. Sue used fencing around your farm as a context to conceptualize and motivate students about perimeter. All the teachers extended and added problems that changed the context of the mathematics the students were learning. For example, Sue changed the sizes and kinds of drinks in the Pepsi problem. She compared the 2 liter pop to a gallon of milk and asked students to use the skills of proportions and equations to solve the problem. This created cognitive conflict because students had to discover how to make the products the same amount for the mathematics to work correctly. The teachers used these contextual shifts in their problems to deepen students' understanding of the concepts and procedures and developed their conceptual schemata for easier retrieval and transfer.

The teachers believed that learning mathematics developed from social and situational contexts. Kathy said, "I'm a real constructivist so I prefer to teach math where they construct their own learning. Where you give them the tools they need and let them just explore and try to solve a problem and get a basic understanding." They wanted their students to know about metacognitive features: what, when, why and how of the mathematics. The teachers taught mathematics contextually by giving students many strategies to choose from and allowed the students to self regulate themselves in different situations and contexts. The knowledge, strategies, and expertise the students learned by making their own choices about when to use what where developed students' understandings of themselves and their social worlds as well as giving them a deeper understanding and conceptual mental frameworks of the mathematics.

Dimension 3: Interactive and experiential learning environment. When mathematics classes started in these classrooms students got out of their seats and started discussing their ideas and experiences with each other and the teacher, manipulated objects to test conjectures, devised their own algorithms and experimented to see if their plans were effective in solving a problem given by the teacher. Polya (1981) quoted Lichtenberg in saying, "What you have been obligated to discover by yourself leaves a path in your mind which you can use again when the need arises" (p. 103). These teachers wanted students to actively experiment and discover the mathematics in the context of solving a problem that enabled them to access and use the information throughout their lives. "It's more they discover, we discover, and they practice, we discuss it, they work with a partner, it's just more engaging for kids and it's more fun for me to teach than just having to stand up there and tell them something and expect them to understand it," says Callie. Sue said, "I like to start out a lot of times by having students discover something so that they can discover the formula rather than just giving it to them and saying this is how you do it."

The teachers knew the pedagogical decision to teach mathematics through discovery and experimentation would ignite student motivation and curiosity. The National Research Council (2001) suggests that children are both problem solvers and problem generators; they not only attempt to solve problems presented to them, but they also seek and create novel challenges. The teachers in this study wanted to build on their students' motivation to explore, succeed, and understand the mathematics they were teaching and that learning could be enhanced by interaction. Senge et al. (2000) states, "human cognitive development involves just as much 'body knowledge' as it does 'mind knowledge.' Learning is inseparable from action. When we assume that learning takes place only in the head, we deny much of what makes us human" (p. 37). Not only is knowledge as much of mind as body but it is also continually changing especially in the mind of the learner. Piaget (1973) wrote that "knowledge is . . . changing. . . . It is not momentary; it is not static. . . . It is a process of continual construction and reorganization" (p. 1-2). The teachers wanted their students to experience mathematics in as many modalities as possible to help them conceptually understand the concepts and procedures deeply so they could transfer the skills and enhance their capability to do something with the information. Bransford, Brown, & Cocking (2000) states that this new science of learning is beginning to provide knowledge to improve significantly peoples abilities to become active learners who seek to understand complex subject matter and are better prepared to transfer what they have learned to new problems and settings.

The teachers in this study wanted their students to take charge of their own mathematics learning. They told me continuously that they wanted their students to be accountable to the mathematics that they were learning. "Discovering it makes them accountable, makes them want to learn more," said Callie. She continued, "If they were told what to do versus discovering it and manipulating to find out their understanding of the answer they would have no buy in it." Students were allowed and encouraged to develop their own algorithms and strategies to solve problems and their work became the focus of instruction during the lesson. Kamii (1989) believed that when students invent their own algorithms they become more competent than if they were told algorithms and the "procedures children invent are rooted in the depth of their intuition and their natural ways of thinking" (p. 14). This created personal connections for students' learning both metacognitively and mathematically. "Allowing students to create their own algorithms helps them link conceptual knowledge with the procedures they select," said Bruning et al. (2004, p. 317). As they try out various algorithms, the consequences (success or failure) of using certain procedures often will result in changes in the conceptual framework (Schoenfeld, 1992). This autonomous process of interactive feedback between conceptual and procedural knowledge leads students to improved representation of problems and increasingly sophisticated mathematics proficiency (Rittle-Johnson, Siegler, & Alibali, 2001). This personal approach to teaching and learning mathematics helped students determine that mathematics was relevant to their lives and was important. Wu (1996) states “a major defect of traditional curriculum is its seeming irrelevance. The result is substantial dropouts in mathematics classrooms across the nation.” Students were engaged, excited and challenged in these mathematics classrooms. The teachers went out of their way to create problems for their students that were relevant to their interests.

The teachers used problem solving formats in their teaching of mathematics. They spent considerable time researching and choosing the problems they used. They used some problems that were a part of their school district’s curriculum. Other problems came from research, conferences, books, professional development, etc. The teachers had collected multiple problems for each concept and procedure in their curriculum. They wanted to be prepared to have multiple ways available to meet all the needs of their students. Kathy said,

> I want them to always figure out a way so that they could be successful in mathematics. They need to see that my brain doesn’t think necessarily the way someone else’s does but there is a way I can do it that is just as correct and so we need to find that way that works for them.

Good problem solving tasks share some common features (Stein, Grover, & Henningsen, 1996). First, they are accessible to a wide range of students yet have no quick solution. Second, they require some amount of investigation or data gathering. Third, there are multiple mathematical paths to a solution or solutions. Fourth, they present opportunities for generalizations to be formed about mathematical relationships. Fifth, they require problem solvers to justify their steps and conclusions based on the givens. And lastly, they allow for sense-making in that solutions and generalizations can be understood by reference to the original problem context. These teachers selected and taught problems that met all these features and more in their teaching. They expected all of their students to progress their current understandings of the mathematics they were teaching. They knew their students were at different levels and used different pathways to solve problems. Students not only collected data but mostly had the responsibility of deciding what data and what structure of that data collection were needed to solve the problem. For instance, in the circumference problem students were asked to measure circular objects in the room testing students' conjectures to determine which if any were correct. This mathematical experimentation led to multiple pathways of discovering the formula for pi. Students' experiential experience with the problem both physically and mentally developed their conceptual framework in ways that will help them correctly retrieve this information at a different time and place.

The teachers purposefully wanted their students to have experience with the concept of the mathematics they were teaching before they would introduce or work with the traditional procedure. This brought out preconceptions, prior knowledge and current cognitive structures in their students' thinking. Callie said,

> I didn't want to just show them an example without them trying it out first. That release of responsibility sometimes doesn't work with math. I wanted to see where they were going to go with it first because sometimes it goes where it's completely the opposite and sometimes you're completely impressed with what they do on their own with no guidance.

Sue said, "I like to start out asking a question and getting them to wonder about something and let them try it out a little bit. Then come back to how we can solve this together and go through what's the mathematical way of figuring this out." Hiebert and Wearne (1986) concluded that if children are taught procedural knowledge before they understand the concepts, it is very difficult for them to link the two together. It is thought that if misconceptions were formed about the content from working and memorizing procedures it would be hard to change that conceptual schema cognitively. The teachers also wanted students to have some experience with the concept so they would have mental structures to relate the procedure to which helps conceptual understanding.

The teachers selected problems so students could discover multiple answers and pathways to a solution. They wanted students to have multiple strategies to solve problems so all of their students could be successful. This taught students that mathematics was a thinking discipline with many paths to correct answers. Current research shows that solving mathematical problems in multiple ways benefits students by developing their creativity, critical thinking skills and fosters advanced mathematical thinking (Leikin, 2007; Leikin & Levav-Waynberg, 2008). Students also benefit by increased mathematical perseverance and multiple solution pathways. Schoenfeld (1983) noted that when students perceive that the problem at hand can be solved or is allowed to be solved in different ways, then this situation increases students' engagement and helps them not to give up working on the problem. The teachers expected and emphasized students' multiple non-routine mathematical algorithms, conjectures, thinking, and solutions to the problems. Kathy said, "But if you listen to these kids there are ten different ways they solve a problem and that is fine, that's great." These excellent teachers selected students' examples of working with the mathematics and the problem that were correct and incorrect to help students make connections and adaptations to their current conceptual and procedural understandings. Bransford et al. (2000) states, "Important ideas in mathematics are developed as students explore solutions to problems, rather than being a focus of instruction per se." By allowing students to autonomously conjecture and explore their solutions they learn that mathematics is a thinking discipline that is connected to problem solving and is not just a series of procedures and formulas to calculate. They develop more complex conceptual schemata connections that enable students to use the mathematics in conditionalized ways. Sue talks about how her students were making connections,

> I think they start to see how this is going to apply to algebra and I make it fun and talk about things like you have to be fair and relate it to kids wanting to be treated fair. It's fun to teach and they start to see how it all works together.

The teachers created and used manipulatives and representations that students interacted with to deepen conceptual and procedural mathematical understandings and to challenge and prove students' conjectures and thinking. Mathematical language was used in representations to visually see relationships of how the mathematics works. Van de Walle (2004) notes, "A rich understanding of number, a relational understanding, involves many different ideas, relationships, and skills" (p. 115). The teachers wanted students to discover many ways to interact with and use mathematical knowledge. These teachers wanted their students to learn mathematics using verbal and visual modes of learning. Sue said, "I like to draw pictures to show a process of the whole. It helps them see it and gives them that model so that they can make sense of it logically." "I think that manipulatives are a great way to have them (students) experience what mathematics is," said Kathy. The teachers used individual boards to acquire visual cues to how students were understanding and processing the concepts and procedures that were being taught.

Mathematics teaching and learning in these classrooms was represented as a live organism. The Webster's Encyclopedic Unabridged Dictionary defines organism as "any complex thing or system having properties and functions determined not only by the properties and relations of its individual parts, but by the relations of the parts to the whole." Students interacted and experimented with mathematics and attempted to find ways to solve problems. Mathematical parts were examined and utilized within the whole solution of the problem. Many different kinds of discoveries were being made conceptually and procedurally about the mathematics with the goal of understanding as much about it, in as many ways as possible. Focus was on making sense of the many relationships and connections it has with itself and other mathematics to understand it more. Students and teachers discussed their thinking and dissected current mathematical work to obtain feedback to the relevancy and truth in an attempt to prove their thinking correct. Teachers adjusted the instruction to meet students' needs. The effectiveness of teaching lies in the ability of the teacher to adjust both content and pedagogical methods to students' prior knowledge, thinking and performance during interactive teaching (Grossman & McDonald, 2008; Hiebert & Morris, 2009; Lampert & Graziani, 2009). Students justified their findings through discourse with each other and the teacher. Yackel and Cobb (1996) believed that discourse about different ways to solve problems enabled the development of students' autonomy. Students were allowed to find their own ways to solve problems as they interacted and explored the mathematics. This improved the classroom atmosphere and enabled students to freely express their own ideas and solutions. This lively environment of exploring and discovering mathematics allowed the teachers to examine the complexity of mathematics and develop the cognitive structures of students' mathematical learning and thinking.

The teachers were continually involved in their interactive and experiential environments. Metaphorically, the teachers were the oxygen of the learning environment. These teachers facilitated and guided the learning environment, assessed students continuously, chose relevant problems, made adjustments and diversions, reviewed prerequisite knowledge, asked many questions, conducted discussions, created manipulatives and representations, provided feedback, presented and used students' work, and made connections and relationships to mathematics and life. Students were actively experimenting with the mathematics, manipulatives, and representations, interacted in discourse with each other and the teacher, created their own conjectures and algorithms and tested them, asked and answered questions, solved problems, and found multiple answers. The focus of the mathematics teaching was on students' experiences and understandings—in other words—their learning.

Dimension 4: Students as capable thinkers. In this dimension I will show how the teachers valued their students' thinking and believed that their students were capable mathematical thinkers. I will also show the benefits students received from these teachers' epistemological beliefs. The teachers in this study focused on students' thinking and understanding of the mathematical concepts and procedures of their curriculum. Throughout each lesson, students were challenged to explain their answers and support their arguments instead of relying on the teacher or a book for verification of correctness. Deb Ball's (1993) goals of mathematical instruction included developing a practice that respects the integrity both of mathematics as a discipline *and* of children as mathematical thinkers. Students had to think to develop conjectures, experiments, build proofs, and to frame and solve problems. This type of teaching and learning resembled the work of a mathematician and was intentionally the way these teachers wanted to teach mathematics in their classrooms. The teachers continuously informed students of the behaviors of mathematicians during lessons.

The teachers taught students mathematics instead of teaching mathematics to students. They focused on students' thinking first and extended the mathematics lesson from that thinking. The teachers taught their students more than mathematics; they taught them different ways to think. Sue said, "I emphasize to them a lot that they need to think about things logically." These questions and many more like: "Why are we doing that? Why does that happen? Why do you put the zero there? Why do we have to do that? Does that make sense? What if it was . . . ? What would happen if . . . ? " were some of the questions frequently asked in these classrooms to get students to think about the mathematics for deeper understanding and to make connections with the discipline of mathematics. These teachers developed thinking processes that gave their students a foundation for developing problem solving strategies. The teachers encouraged students to look for patterns and relationships through open-ended problems. They helped students develop number sense by emphasizing numerical relationships and connections in different contexts. The students learned to develop models of their thinking through drawing pictures, interacting with representations, using manipulatives, and to analyze their thinking for sense making and logic. These experiences helped students construct problem-solving methods for thinking about ways to find solutions to problems. The teachers taught their students that mathematics was more than producing answers; it was about learning to think.

The elementary teachers in this study believed that their students were capable of deep mathematical thinking. Kathy said, "I like to get into their heads and see what they're thinking. It is adamant that they understand what it is they're doing because I didn't." The teachers focused on the students and their processing of mathematical understandings and then developed mathematical concepts and procedures to progress their cognition and thinking. The National Research Council (2001) believes that,

> Children lack knowledge and experience but not reasoning ability. They refine and improve their problem-solving strategies not only in the face of failure, but also by building on prior success. They persist because success and understanding are motivating in their own right.

Theories of cognitive development (e.g., Bruner, 1972; Gardner, 1991; Piaget, 1973; Vygotsky, 1978) emphasize children as active learners who are able to set goals, plan, assemble, organize, and revise information as well as developing strategies of remembering, understanding, and solving problems. These teachers helped students develop their own capacity to think critically by directing students' attention, structuring their experiences, supporting their learning attempts, and regulating the level of difficulty and complexity of the concepts and procedures they were teaching. They were able to achieve these tasks through their assessments, both formally and informally, of their students' thinking. Sue said,

> You really have to listen to kids because a lot of times what you think you said and what they understood are different. By listening to them and having them talk about it helps them to develop a better understanding of it. They can't just keep sitting there taking in the information. They need to verbalize it back to really understand it.

Kathy said, "I've got to know where they're coming from and I got to know what they are thinking because if they're wrong it doesn't matter if their thinking is good. I just want them to tell me what they're thinking." Callie said, "I think they feel that their thoughts are validated if they're addressed if they bring them up." The teachers showed their students that mathematics was a discipline about thinking and that their thinking was valuable. Students gained confidence to construct mathematical knowledge that was powerful and correct.

The teachers affirmed their students' thinking by repeating what they said and using their ideas and dialogue to guide the lessons. They valued students' approaches to mathematical thinking and recognized their discoveries in exploring other ways of thinking students were choosing to use and how they chose to use them. The teachers listened closely to their students and repeated their responses to other students and used that thinking to develop understanding and made connections to mathematical concepts and procedures. In this way, teachers validated the power of their students' thinking. This built students' confidence in their thinking capabilities and beliefs in their own inherent value as human beings. These teachers focused on the inherent value of their students as they showed them they could be successful at mathematics or what ever they wanted to become. It became clear to the students in these classrooms that grades were a nonissue and were more interested in what they were doing than with how they were doing. Student motivation became more about reaching a larger goal of finding solutions and making contributions to mathematical experiments and discoveries. Senge et al. (2000) believes that, "all human beings have the capacity to generate novel, original, clever, or ingenious products, solutions, and techniques if that capacity is developed" (p.201). These teachers developed the capacity of their students to be creative, empowered, valuable, and successful. The teachers focused on the students and their specific needs and allowed them to take charge of and create their own learning. Because the teachers believed their students were capable thinkers they created environments where students became active participants rather than receivers of information.

The teachers wanted their students to be responsible for and accountable to the mathematics they were learning. They purposely asked students to solve problems on their own without instruction so that students could interact with the mathematics in an attempt to find solutions and create personal algorithms from their own prior knowledge and misconceptions. They asked students questions, made representations, and had students work with manipulatives to solicit mathematical thinking and asked additional questions to guide students thinking in discovering solutions and making connections on their own. Sue said, “I keep asking them questions of what we did before so that they can figure out on their own what the process is and why.” The teachers created situations and expected the students to use their own thinking to solve problems and share their ideas with each other and the class. The teachers used explanations of students’ thinking to build their lessons and make connections to concepts, procedures, and real life. This process allowed students to be active participants who were accountable for their learning. Callie said, “Kids can be teachers, too. Giving kids tons of opportunities to explain so that kids can learn from other kids not just from you. Kids need to communicate and talk about it and not just show it on paper.” The teachers in this study empowered and motivated students to want to learn, explore, and discover mathematics. The teachers guided students in proving their own answers and did not establish themselves as the authority for correctness of thinking. Instead, the teachers redirected students by asking additional questions, developing representations, or presented another point of view for students to test hypotheses or conjectures and validate and improve their own thinking. Students developed autonomous behaviors by creating and evaluating their own thinking and learning. The students considered themselves as having the primary responsibility for their learning by creating, thinking and communicating mathematics with each other guided by the teacher's supervision. Kamii (1989) noted that when children exchange points of views with each other they develop autonomy. Piaget (1973) believed that autonomy should be the goal of education. These teachers also developed autonomy to build students' confidence and inherent value of themselves.

The teachers emphasized students' thinking in every phase of their mathematics teaching. Students realized their thinking was valued and important. They knew the teacher would solicit their thinking, carefully listening to and validate what they were saying, and respectfully use their thinking in class to learn more about mathematics. The teachers learned what their students knew by asking questions that required them to think. Sue said, "I just had to keep asking questions because I think if you ask questions then you get kids to keep thinking about what do I need to do next or I ask more general questions so that they can keep thinking. I like to ask a lot of questions during class so that I can see if they're understanding." Fosnot (1989) wrote, "asking questions about thinking causes people to think more" (p. 88). They asked students to model their thinking on individual boards, white boards, overhead projectors, manipulatives, representations or any way they could to externalize their internal thinking processes for the class to observe and interact with. Sue said, "By putting them in groups and listening to what they're doing and saying and discussing helps me see where they maybe have a misunderstanding." The teachers had their students work with materials and boards to bring out their conceptual thinking, prior knowledge, and misconceptions. Lampert (1989) noted that if students could represent concepts through materials it contributed to long-term understandings. By using multiple representations of a mathematical concept or procedure the teachers helped students make sense of mathematics.

The teachers also taught their students functional thinking. Functional thinking is defined by Smith (2003) as “representational thinking that focuses on the relationship between two (or more) varying quantities” and for which functions denote the “representational systems invented or appropriated by children to represent a generalization of a relationship among quantities.” Research increasingly shows the ability of elementary students to engage in algebraic reasoning in ways that were previously reserved for older students (e.g., Bastable & Schifter, 2003; Blanton & Kaput, 2003; Dougherty, 2003: Schifter, 1999). Sue’s proportion problem episode is an example of teaching and learning functional thinking that prepared students for advanced mathematical study in algebra. Sue said,

> I think they start to see that is going to apply to algebra and we talk about how you have to be fair and relate it to kids and they want to be treated fair, what you do to one side you have to do to the other, and how the variable has to be all by itself. It’s fun to teach and they start to see how it all works together.

The teachers made functional connections and relationships between variables and numbers using representations, manipulatives, and discussions. Sue’s use of different colored markers on whiteboard representations are an example of teaching functional thinking.

The teachers encouraged reflection and helped students extend their thinking about mathematical concepts and procedures. They directed students’ attention to look at a problem deeply to explore their own understanding of the mathematical processes and concepts rather than directing students to correct and incorrect answers. Students knew that the teachers wanted them to know the processes they used to get to the answer more than the answer itself. The teachers required students to show their work and explain how they got answers. These teachers helped students extend their thinking about a mathematical concept or procedure by presenting a variety of solution strategies and applying what they learned to new situations and contexts. Reflective questions were asked to guide students to reflect on their own thinking. According to Duckworth (1987), it is by thinking and reflecting about their thinking that students get better at thinking. She added if students do the work themselves, reflecting and making connections, it is more likely that students will be able to repeat what they have learned. The seven divided by four problem is an example of students working on a problem in pairs conjecturing, reflecting, and making connections to fractional understandings by themselves. Students were allowed to struggle on a problem as they tried to make sense of the mathematical concepts and procedures on their own with reflective guidance from the teacher.

The teachers believed the skill of thinking was a valuable educational goal for their students and mathematics was a subject that could enhance students' ability to think. Howard Gardner (1999) believes, "the most important and irreducible purpose of education is to help students better understand the major disciplinary ways of thinking. This means establishing ways of thinking in students that they haven't experienced yet: teaching them what it means to think scientifically, historically, artistically, ethically, and mathematically."(p. 19). These teachers wanted to foster multiple ways of thinking and using strategies about mathematics in their students. They also wanted students to explore and discover their own experiences to guide their learning. Gardner stated, "One of the few things we've clearly established in cognitive research is that, in the absence of sustained inquiry, people develop all kinds of misconceptions that make it impossible for them to think scientifically." He went on to suggest that students should, "grasp a number of different models of ways of representing their knowledge. You can't understand something if you only hold *one* model of it. An expert is a person who has lots of models of a field of study – lots of ways of thinking about it." The teachers in this study took great effort to model and present students with problems and solutions with multiple pathways of experiencing mathematical concepts and procedures. This afforded students opportunities to learn multiple problem strategies for different pathways to success. Kathy said, "They [students] need to see that my brain doesn't think necessarily the way someone else's does but there is another way I can do it that is just as correct. So we need to find a way that works for them. I want them to always figure out a way so that they could be successful in mathematics."

Students benefitted in other ways from the teachers' beliefs that they were capable and responsible mathematical thinkers. The teachers facilitated answers from students that were powerful and successful in building students' confidence. For instance, Sue noticed a student's discovery of using "all" as the denominator in the probability problem and had him come up to the board to share his thinking with the class. His thinking was correct but had to be developed in conception through discussion and manipulation of the representation which developed all the students' functional thinking abilities. The important thing here was the student's thinking was exceptional and valuable for the class to proceed in understanding the problem. Students were excited about mathematics and sharing their ideas and wanted to learn more because they were valuable thinkers with important contributions. The teachers created situations so students could examine the complexities of the mathematics which helped them become critical thinkers with multiple perspectives and strategies for solving and thinking about problems. Students also developed metacognitive skills for learning and thinking. As they explored problems, they tested conjectures and hypotheses for validity and made connections to other mathematical concepts, procedures, and how to apply what they have learned to other contexts and real life. The students were allowed to take command over the mathematics and their learning. They created it, taught it, explained it, understood it, and represented it in multiple ways.

Dimension 5: Participatory communication. I used the terms social interaction and communication simultaneously in this dimension as I describe the social environment of these teachers' mathematics lessons. Social interaction describes activities, such as data collection, that may or may not include verbal communication. I consider both social interaction and other forms of communication as participatory communication in this dimension. Social interaction was continually occurring in these teachers' classrooms through communication between the teacher, student, other students, and the mathematics. The communication contained both verbal and visual interactions between the members of the classroom. When I asked Kathy how important communication was in her math class she said,

> I've talked about communication a lot because I think it is the most important thing. Not only how I communicate to them to teach the skill but how they communicate to each other and back to me so that I know they have that understanding. It doesn't do any good to know it if you can't explain your thinking. They have to know how they did it. They have to know what they were thinking and be able to tell somebody that. Mathematicians talk all the time.

Each member, students and teachers, of these classrooms took on the roles of listener, speaker, and analyzer equally at various times. Verbal communication occurred in pairs, small groups, discussions, through interacting with representations and manipulatives, games, and exploring processes and solutions to mathematical problems. Presenting personal conjectures and algorithms, working with games, manipulatives and representations, collecting and sharing data, discovering multiple solutions to problems, and asking and answering questions were tasks teachers used to enhance opportunities for students to participate in communicating and processing mathematical thinking.

The teachers created positive social environments for their students that allowed them opportunities to explore their worlds, take risks and learn. Each student was required to participate and interact with the mathematics, other students, and the teacher. They received individual sets of manipulatives to work with and worked with individual boards showing how each student was thinking and what they did while interacting with the mathematics for the class and teacher to examine. The teachers assembled students into small groups or pairs to explore, discuss, interact, and share what each was thinking. This ensured each student had responsibility to participate and communicate their own thinking. Bruning et al. (2004) suggests teachers decentralize discussions to foster cognitive growth. The teachers purposely chose small group sizes to maximize students' opportunities to participate, contribute, and develop their independent thinking cognitively. I noticed some students were more comfortable sharing their mathematical ideas with a peer or small group and would offer more dialogue when they had opportunities to turn and talk with each other. "Some students are reluctant to take part in a full-class setting because of perceived lack of knowledge or shyness," says Bruning. Bruning and his associates state, "Classroom discussion can be seen as the everyday expression of the idea that students are active agents in their own learning, enabling students to construct new conceptions and acquire new ways of thinking" (p. 204). They go on to proclaim that social interaction is fundamental to cognitive development. The excellent teachers in this study provided environments that supported students' knowledge development of self-awareness and self-direction through social interaction and communication. Their teaching and learning of mathematics was highly social with students interacting with each other and the teacher, and physically with objects and representations to construct personal meaning from various activities. The social interaction for these teachers' students was a natural and safe way for them to learn mathematics. Senge and colleagues (2000) believe, "We are social beings. We congregate in groups, find being listened to therapeutic, draw energy from each other, and seek reciprocity" (p. 204). William Isaacs, founder and director of the MIT Dialogue Project (1999) found,

> During the dialogue process, people learn how to think together – not just in the sense of analyzing a shared problem or creating new pieces of shared knowledge but in the sense of occupying a collective sensibility in which the thoughts, emotions, and resulting actions belong not to one individual, but to all of them together. (p. 357)

These teachers used participatory interaction in their mathematics classes to bring a sense of shared experience as the students explored, discovered, and shared their own experience with their classmates as they learned together. The students discovered they were capable of thinking and constructing knowledge on their own and making contributions to the class as a whole. Working together in various forms of small groups required students to justify their ideas and test the feasibility of their solution strategies on others. Chinn and Waggoner (1992) suggested that when students share alternative perspectives, they give their personal reactions and interpretations and consider the viewpoints of other participants. The teachers also chose this pedagogical strategy to enhance social skills students need to be able to work effectively with other people. They included listening, consensus seeking, giving up an idea to work on someone else's idea, empathy, compassion, leadership, and knowing how to support group efforts. These skills were taught by these teachers in addition to the mathematics curriculum during social interactions and group work as students attempted to solve problems together. These teachers performed many important roles during participatory communication without taking a dominating role. Some of the roles the teachers chose were organizers, questioner, observer, assessor and participant.

The teachers' ultimate goal for their students in mathematics was conceptual understanding of mathematical concepts and procedures. They chose social interactions and communication as a pedagogical strategy to develop students' cognitive structures of mathematical understandings. Cognitive research has shown that social-cognitive activities, such as well-managed cooperative learning and classroom discussions, stimulate learners to clarify, elaborate, reorganize, and reconceptualize information (e.g., Calfee et al., 1994; Cowie & van der Aalsvoort, 2000). The teachers wanted their students to verbalize their thinking to find out how their brain was processing what was being taught and what information was retained. Kathy said, "Putting it into words and being able to verbalize it is vastly different from just being able to do it." When asked about group work Sue said,

> I think they have to do it. They can't be passive listeners. I started to see that by putting them in groups and having them explain it to each other then they were using vocabulary and had to repeat what they had learned. So by having to repeat that and explain to somebody else they were starting to have a better understanding. By them working in groups and having to explain it to each other

> and to see where their errors were, their mistakes, then I think they started to understand it a little better.

Participatory interaction and communication helped the students organize a complex web of conceptual understanding that included semantic understanding and schemata building. Active participation in these classrooms included physically interacting with manipulatives, representations and objects as well as social verbal communication of personal understandings with others. In this way the mathematics and the mind of each student came alive actively recreating itself as it was being interacted with. These teachers believed young children were capable of complex reasoning, problem solving, and sense making abilities. It is believed that students can start learning to justify their mathematical ideas in the earliest grades in elementary school (Carpenter & Levi, 1999; Hiebert et al., 1997; Schifter, 1999). In all of these teachers' classrooms students talked about the concepts and procedures they were using and provided good reasons for what they were doing every day in class. According to Maher and Martino (1996), students need to be able to justify and explain ideas in order to make their reasoning clear, hone their reasoning skills, and improve their conceptual understanding. The teachers in this study expected students to verbalize their current understandings of mathematical concepts and processes. Kathy stated,

> They really don't know what their brain is doing to help them solve a problem. They were using the strategies often but the couldn't explain it and mathematicians have to talk. You have to talk mathematically and I need to hear what they're thinking is. If I don't hear it I assume they don't know it. They don't have a deep understanding. I always want to know what they've retained and what they can verbalize. If they don't know how to explain it they probably don't know it. So I see talking math and understanding go hand in hand.

The teachers made explicit differences between students being able to execute problems and being able to explain the concepts and processes they used verbally. They also wanted students to articulate their mathematical thinking using precise language. The teachers wanted the students to learn to use their minds and become very specific at expressing their thinking. In the circumference and perimeter problem, Sue asked a student a number of questions in an attempt to get him to articulate his thinking precisely so he and his classmates would have a clearer understanding of the definition of perimeter. She was adamant about the language the student was using and that it made his dialogue confusing in trying to understand the precision and depth of his thinking. When describing their thinking, students must be precise and explicit in their talk, especially providing enough detail and making referents clear so that the teacher and fellow classmates can understand their ideas (Nathan & Knuth, 2003; Sfard & Kieran, 2001).

The teachers expected their students to go beyond just providing answers. They asked students to describe how they solved the problem and why they proposed specific strategies and approaches to solving it. Each teacher was adamant about precise mathematical understandings and explicit verbal explanations of their students. These social interactions between the teachers and the students created an environment where students constructed new knowledge, acquired habits of reflection, increased metacognitive knowledge, and developed conceptual webs of understanding. Higher mental functions develop through a process by which the learner internalizes and transforms the content of social interaction (Fall, Webb, & Chudowsky, 2000). Students also benefitted from participatory communication because they were at differing levels of mathematical understanding and skill. This pedagogical strategy allowed students to communicate what they did, why they did it, how they did it, and shared their thinking with each other to develop at their own pace and still be successful. According to Yackel, Cobb, Wood, Wheatley, and Merkel (1990), children at differing conceptual levels use different solution methods and interpret tasks in different ways. Students were able to hear multiple verbal explanations from other students and the teacher during dialogue. They were able to question and listen to each other until they understood each others strategies. This allowed opportunities for each student to find an explanation that made sense to him or her. Callie said, “Students can be teachers too. Sometimes they can understand another student’s explanation over mine.” Bruning et al. (2004), suggests teachers assist students in verbalizing, and if possible, visualizing processes used in solution attempts and noted that group processes are very useful for developing flexible mathematics thinking and positive attitudes toward mathematics. The teachers wanted to give their students many opportunities to understand mathematics in multiple ways which in turn expanded their conceptual and procedural knowledge and skill. Describing, explaining, and justifying one’s thinking all help students internalize principles, construct specific inference rules for solving problems, become aware of misunderstandings and lack of understanding (Chi, 2000; Chi, Bassock, Lewis, Reimann, & Glaser, 1989), reorganize and clarify material in their own minds, fill in gaps in understanding, internalize and acquire new strategies and knowledge, and develop new perspectives and understanding (Bargh & Schul, 1980; King, 1992; Rogoff, 1991; Webb, 1991).

The teachers created a questioning culture in their mathematics classrooms and used questioning techniques in various ways to develop students’ conceptual and procedural mathematical understanding. These teachers asked questions continuously in their math lessons. They used questions to inquire, assess, guide, and explain students’ thinking and asked cognitive conflict questions to have students re-think their reasoning. Callie said,

> It's more interactive and it's okay to ask questions because I ask questions right back. The questioning is so important in math and reasoning. If I ask kids questions it really helps me assess them and whether they are understanding or not. They ask me questions and I can see how deep those questions are or where they're coming from to ask those kinds of questions.

The teachers were always walking around listening to students and asking questions so they could understand what their students understood and what was needed to develop that understanding. Kathy said, "I have to walk around, I have to see what they are doing, I have to ask them questions. I'm continuously trying to see where their understanding is and what piece I need to put in there to give them more understanding."

I was interested in what questions these excellent teachers asked and how they asked them so I decided to analyze a teaching episode for one of the teacher's questioning strategy. I chose to use Sue's teaching episode on proportions because of the length of the episode and the amount of questions asked to a specific student and questions asked of other students in the same sequence. This questioning strategy is considered typical in any of the three teachers' classrooms at any given time. I used this episode to look specifically of what questions were asked, what sequence was used and why they asked a particular question. I used Megan Franke and her colleague's (2007) definition of types of questions asked in my analysis. They categorized questions into five kinds of questions:

> General questions - questions that were not related to anything specific a student said
> Specific questions – question about something specific a student said
> Probing sequence of specific questions – more than two related questions about something a student said

Bundles of questions- at least two questions and did not provide the student opportunity to answer any of the questions.
Leading questions – specific sequence of questions that provided opportunities for students to respond by guiding students to a particular question or explanation

Mrs. Johnson: There are two face up for every face down card. If six cards are face up how many are face down? So, how can we set that up as ratios and proportions, Ted? First of all what is going to be our words?

In this sequence of questions, Sue asked a general question followed by two questions that were leading questions to help guide students in what they needed to do to accomplish the task. All the teachers asked cognitive processing questions to help students metacognitively make sense of the process or concept being learned. She asked the first two questions as bundles of questions that students did not have opportunity to answer. This was intentional because she asked the questions to focus students' attention to a process and the third question was the beginning of that process.

Ted: Face down cards and face up cards.

Mrs. Johnson: Okay, face down and face up.

The teachers always revoiced the students' responses to questions. Studies have found that often revoicing supports the development of mathematical ideas (Forman & Ansell, 2002; O'Connor & Michaels, 1996; Strom, Kemenya, Lehrer & Forman, 2001). In this episode, Sue drew a visual representation on the board of the student's response.

Mrs. Johnson: I don't try to write all the words. Remember, I said that mathematicians don't like to write so I'm going to use abbreviations. So, what do I know?

The dialogue the teachers used before they asked a question or series of questions informed the students about how to do something or why something is done in a particular way. Notice that the question being asked here gives the responsibility of the intellectual work to the students to figure out what is needed.

> Ted: There are two face down cards for every face up card.

> Mrs. Johnson: So, what do I put here?

Again, the teacher asked a specific question that puts the responsibility of the intellectual work on the students. She is asking the whole class the question as one student is currently providing answers.

> Mrs. Johnson: Okay, and then it says what?

All the teachers modeled strategies like this for students to learn a process through a questioning culture that helped students cognitively tackle a problem.

> Ted: That there's every, for every sixth card, for every six cards up.

> Mrs. Johnson: Does that go here?

The teacher was using a representation to simulate the process of how to set up a proportion problem and to visualize mathematically what was being said. At this point the representation led the student to proceed without any more questions from the teacher.

> Ted: Yes! So, so you got to figure out how you got from the one to six. So you have 1 x 6 = 6. So then you take 2 x 6 to get the other.

Sue completed the representation to depict the reasoning of the student for the whole class to see as he was explaining it.

> Mrs. Johnson: Okay, so how can we check our answer, John?

Here the teacher asked a general question to a specific student to metacognitively express to students that it is important to check your mathematical answers by providing a proof. All the teachers frequently would intentionally ask a specific student a question. There were many reasons for this strategic move that include; attentiveness, unique student thinking, assessment, acknowledgement and others.

John: You can take 2 x 6 and 12 x 1.

Mrs. Johnson: So 2 x 6 is 12 and 1 x 12 is 12 so that confirms that our answer is correct. Okay, that one was fairly simple, but again, we put our labels so that we know what is what. I want you to try two or three on your own. Now, I'm going to caution you because sometimes were talking about all cards and not face up or face down and that's when it gets to be a little confusing.

Here the student's answer is revoiced and the contribution was completed precisely for the student and the class to acknowledge showing that details matter in mathematics. The students then worked independently on a couple of problems. Sue brought the class back together after a few minutes and intentionally asked a specific student to put her answer to the first problem on the board for the class to examine. She asked this specific student to present her answer because she knew that it would expand the students' understanding of setting up proportions properly and would create cognitive conflict to the incomplete strategy that many of the students were implementing. The teachers in this study knew what to focus on to pursue students' thinking and improve their understanding of the underlying mathematical concepts and procedures being taught.

Mrs. Johnson: Okay, I don't want just an answer, I want to know how you got there. I want you to do just like I did. I want to see your labels, your numbers, and your computations.

The student put her answer on the board:

$$\frac{U}{D} = \frac{3}{5} = \frac{12}{20} = 20 \text{ cards}$$

Mrs. Johnson: Does everyone agree with what she has there?

This was a specific question asked of the whole class about the girl's answer. It stirred up lively conversation about the work with multiple students agreeing and disagreeing with different aspects of the answer. The teachers asked questions, such as this, continuously to create cognitive conflict for the students. These kinds of questions helped the students rethink their thinking and were designed to help students experience conflict with their inaccurate thinking so they could accommodate more accurate and correct knowledge. These teachers wanted their students to logically change their minds about an answer by learning why their answers were incorrect and would not fit into the new conceptions that were being developed cognitively. The teachers wanted students to understand why things are instead of being the authority that tells students whether they were correct or not. This process showed students they were capable thinkers that could develop their own thinking to deeper conceptual understandings of the world around them instead of being told what to think. Simon and Schifter (1991) explained that construction of new understanding is elevated when students experience disequilibrium and must mentally modify their present knowledge to assimilate their new experiences. These teachers probed students thinking and wanted them to question and examine their own thinking.

Mrs. Johnson: Let’s look at the labels. Three out of every five cards are face up. Sue does not ask a question here but created plenty of conversation among the students about where to put their intellectual focus. Some of the students knew where the problem was from the “labels” clue helping students develop metacognitive skill.

> Mrs. Johnson: It says 3 out of 5 are face up. So, 3 are face up and that 5 should be total. So, it didn’t say 3 up for every 5 down did it. It said 3 out of every 5. So 5 was total. So that is where we sometimes get into trouble because it was just like the problem with rain yesterday. It said 2 out of every 5 days it rains, how many times does it rain in April? We had to know how many days in April there were.

The class worked on another problem and asked if they could draw a picture for the next problem. Sue allowed the students to explore the mathematics and guided her students focus and direction without telling the answer.

> Mrs. Johnson: I assumed that you would be able to do it on your own. Okay, Fred go up there and the rest of you need to look at what he is doing to see if you agree. Fred, if you can explain what you're doing along the way that will help. Labels come in real handy here because they can help you see your mistake.

The boy wrote his answer on the board and the class started to analyze and discuss his work. Sue interrupted the conversation and redirected the students' focus on the specific mathematical idea that was being pursued.

> Mrs. Johnson: The question was how many face up cards are there. He is right to say that there are 6 face up cards. There might be a better way to explain that. Michael, how did you do it?

Sue knew what error the students were making but didn't want to tell them. She wanted them to discover their error in thinking personally. She asked a specific question to a specific student about how he did the problem. Through observation she knew that his thinking would contribute to the conflict the class was experiencing.

> Michael: I had 2 face up cards for every 5 face down cards. So I figured that I had seven and kept the two face up cards. Two plus five is seven.

> Mrs. Johnson: Okay, show that on the board.

As Michael wrote it on the board the rest of the class became more confused and asked "What?" Michael crossed a line through the five on the bottom of the proportion and added a seven.

> Mrs. Johnson: If you change that to seven what else are you going to have to change?

This is a specific question that was related to mathematical relationships of numbers. It was a question that no member of the class was thinking about or could answer. Sue extended their thinking.

> Mrs. Johnson: I agree that it would be okay to change that to seven and I know where you got the seven but I'm not sure that everyone else does because if you're going to change that seven you have to change something else.

A student incorrectly guesses what else should change by asking his own question as a couple of students agree with his answer.

> Student: You have to change the top, don't you?
>
> Mrs. Johnson: Well that's true in some situations but over here there is a reason why he changed that five to a seven.
>
> A student: He added 5 to 2.
>
> Mrs. Johnson: And why did he add the 5 to the 2? Michael, do you know why you added the 5 to the 2?
>
> Michael: Because you can't have.. you have 2 up and .. 5 down for seven.
>
> Mrs. Johnson: So you have seven what?
>
> Michael: Total

The teachers used participatory communication and questions to elicit proper articulation and completion of students thinking which developed cognitive mental frameworks that organized deeper understanding of an idea. Sue helped students develop a web of conceptual understanding using a framework that students already knew.

> Mrs. Johnson: The reason he wanted to do that is because the question asked If you have 21 cards in all how many cards would be face up? So this is one example of when we first started talking about ratios. We talked about how, for example, when I said what's the ratio of boys to girls in our class? The ratio was 6 to 3. But what if I said what is the ratio of boys to all the kids in class? Then it was 6 to 9. This is kind of what we do here. Up to down was 2 to 5 but you needed to find the ratio of up to all cards so

> that you could answer the full question. That is where trickiness comes in reading and solving proportion problems.

The teachers were always trying to make connections between the procedure and concepts of the mathematics they were teaching. This helped students understand better what they were doing and why they were doing it. This communication process included specific questions to students about how they were conceiving the mathematics revealed, in this episode, the students got the correct answer to the problem but were incorrect in their conceptual and procedural thinking in solving the problem. This would lead to incorrect thinking about comparable problems in the future if understanding was not pursued and cognitive conflict not experienced. The intentional sequences of questions asked by these excellent teachers revealed what students actually know and did not know and identified misconceptions students had about the mathematical concepts and procedures being taught. These teachers asked questions that created cognitive conflict where students had to rethink strategies and understandings they initially had in order to advance their thinking. They asked questions that elicited students' explanations and strategies.

These teachers were masterful with asking follow up questions to pursue and develop students thinking about a mathematical idea. The teachers told me their questioning strategies just happen as they are teaching because they were determined by what their students were saying and doing as they interacted with the mathematics. That is one reason why it was so important for the teachers to know what the students were thinking all the time because students' thinking guided instruction and what questions to ask to deepen mathematical understanding.

The teachers also benefitted students using participatory communication because of the enhanced opportunities for feedback and reflection. In these classrooms, feedback and reflection were inseparable and informed both the students and the teacher. Teachers received feedback from students about what they know and how they were attempting to solve problems both visually and verbally. Students and teachers were receiving feedback from the dialogue and questions that were asked. The feedback required students to reflect on what they were doing and what they were thinking as they interacted with the problems. It required the teachers to reflect on their effectiveness of their strategy and make immediate decisions on how to progress from that point in questioning, dialogue, and tasks to further students' mathematical understanding. Rosemary Callingham (2008) determined, "In essence, successful teaching and learning is about dialogue and feedback." The teachers in this study set up educational environments that engaged students in active dialogue about mathematics and provided students with feedback based on what they were saying and doing. This participatory communication allowed all the students to participate at their own level of understanding. The teachers created tasks or activities to engage students in productive dialogue that could be developed in different formats and grow in a variety of ways. Callingham claimed that two essential pieces of information must be provided to students for feedback to be effective: affirmation of what they can currently do and what they need to do next to improve their understanding. These teachers allowed the students to act upon the feedback they were given to empower themselves to develop deeper understandings and discover their own incomplete thinking. Sue said, "When we do group work if you don't agree you have to talk about it. How did you do it? In that process they may discover that as I'm explaining it they may go oh, here is where I made my error." The teachers modeled the process or concept the class was exploring using student responses so students could match their thinking with the teacher's model and discover their own mistakes.

The teachers provided feedback and asked questions to have students reflect on their thinking. This reflection process allowed students to examine their own thinking and determine where incomplete understandings or misconceptions were enabling them to deepen their mathematical understandings. Sue said, "For me their reflection has to be more in their group work and that's where I understand more about the kids. It's a way of understanding their thinking a little bit more." The teachers used feedback and reflection reciprocally to further students understanding of their own thinking and identified what students know and what they need to do to expand and develop understanding. The National Research Council (2001) determined that feedback is on understanding extremely important and that students' thinking must be provided and visible. "Feedback is most valuable when students have the opportunity to use it to revise their thinking as they are working on a unit or project," says Bransford et al. (2000). The teachers in this study used participatory communication focusing on students' work and explanations to teach mathematics for understanding. They had students experience mathematics personally by using students' thinking, conjectures and solution attempts to cognitively develop mathematical understanding and skill.

Dimension 6: Teachers as active knowledge seekers. The teachers in this study were actively involved in mathematics education as teachers and learners outside the classroom even though they have taught for decades. Sue has taught for 32 years, Kathy for 37 years, and Callie for 8 years. It was astonishing to me to find that even though two of the teachers were moving toward the end of their teaching career they continuously talked about plans to obtain more education and knowledge for the teaching of mathematics. The teachers in this study were continuously involved in professional development and research, attended conferences, workshops and college coursework, taught mathematics methodologies to pre-service and in-service teachers, and were mathematical leaders in their school districts throughout their careers. All three teachers have earned Master's degrees in education with as much coursework as possible in mathematics. None of the teachers had a Bachelor's or Master's Degree in mathematics.

All the teachers were involved in continuous professional development activities to improve their teaching pedagogy and disciplinary knowledge in mathematics. Sue and Kathy told me that they have committed themselves to at least one professional development activity each year and have been doing that for decades. "Effective teaching requires continually seeking improvement. Opportunities to reflect on and refine instructional practice – during class and outside class, alone and with others – are crucial in the vision of school mathematics outlined in NCTM's Principles and Standards" (2000). The National Research Council (2001) found that, "Professional development in mathematics needs to be sustained over time that is measured in years, not weeks or months." They also concluded that studies have shown that short-term fragmented professional development is ineffective for developing teaching proficiency.

The teachers specifically chose the kinds of professional growth activities based on what they were interested in and what they wanted to know more about in specific curricular areas they taught in their classrooms. They chose programs that helped them understand the mathematics they taught, how their students learned that mathematics and how to facilitate learning. Kathy said, “I’ve been kind of exploring math facts this year and how kids learn math facts and whether they understand the math facts or not. What’s the best way to teach them?” Sue said, “I still continue to take classes. This summer I plan to help teach classes at the college.” This team teaching experience was a part of her Master’s degree program designed to develop leaders in mathematics education and to share the knowledge she had learned with other practicing teachers. Kathy said,

> The class that I’m taking right now we’re heading in the direction of number sense. How do you get it? What is it? Where does it come from? How do you give it to someone that doesn’t have it? I don’t have the answers to that so I have to find out because I have kids that don’t have number sense.

All the teachers took advantage of professional growth opportunities offered by their school districts and educational service units as well. Sue talked about her professional development goals, “I’m always interested in attending workshops where I can learn more or where I can add to something I have done or maybe it’s a different technique.” Callie determined that her development and growth as a mathematics teacher came through professional development and being on different committees that included a program called “math toolbox” where she learned to understand mathematics, research, and “best practices” for students to learn mathematics. The teachers looked outside of their school districts for professional development opportunities through organizations like their state mathematics association, the National Council of Teachers of Mathematics, and local colleges and universities.

All the teachers were heavily involved in mathematical research as they continually advanced their skill and knowledge. They looked for professional development opportunities that were effective for their specific curriculum and pedagogical style and to develop a deeper understanding of the mathematics they were teaching. The teachers went beyond their school districts' curriculum to find additional information and multiple ways to present and understand information from professional books, the internet, and from colleges and organizations' coursework and workshops. They read professional books on mathematics teaching and learning and mentioned Marilyn Burns as their favorite author because her approaches to the teaching and learning of mathematics matched the teaching and learning philosophies of the teachers. Kathy said,

> I had people like Marilyn Burns show me different ways to use use manipulatives and what those maipulatives show you and how you can get kids to think about that and how you can see what they're thinking and why we're using those hands-on materials. I read a lot of professional books and its been this way for years. I read math kinds of books on what kind of games work, what kinds of activities work, and how kids think. I try to kind of find out what is going on in their [students] heads so I try to read books that will give me information on how kids think.

The teachers continuously conducted research to find additional ways to present information to their students. Callie conducted mathematical research and presented findings on best practices to peers in her school as her school's representative. They were seeking better ways to help all of their students understand the mathematics better. Sue described her research,

> I'm willing to go back and look for lots of things, lots of resources and I have all kinds of extra books around my classroom because I want to find different ways to present things. I put in many many hours after school or at home because I'm trying to find something that will work just a little bit better rather than just using what was in the particular textbook that we got. It either helps me to present it better or because I think the students are going to understand it better. I make two or three different ways to show it because I looked it up in the resources.

All the teachers researched the mathematics they were teaching to obtain multiple pathways of teaching a mathematical concept or procedure so that all of their students could be successful and understand the mathematics that was being taught. Most of the research from these teachers included searching for current teaching strategies and additional resources for improved instruction and filtered research that had been collected by instructors and administrators. Although, Sue conducted action research using group work in mathematics and all the teachers researched “best practices” in mathematics to instruct peers. The teachers were also positioned in professional communities where they were able to interact with other educators and researchers about mathematical ideas and pedagogy. Sue said, “You need to be willing to admit you’re wrong and to change if something is not working or find a different method or way of presenting the information to students. I’ll go and find a different resource that can explain it in a different way.”

The teachers were not afraid to admit they struggled to understand or know something about mathematics. They were determined to find the answers they did not know or completely understand through research. One of the ways they accomplished this task is through the internet. Kathy stated, “If I’m struggling with something then someone else must be struggling with something and we need more information. So you can go online and you can find things all over the place.” When Sue was struggling with work in her Master’s program she said, “I had to use the internet sometimes to go back and check like on Math Forum and work backwards and figure out how they got it or I had to read the books really well and figure out what I need to do.” These teachers also obtained knowledge and skill by attending conventions, teaching classes, and interacting with peers. Kathy attended conventions on the local, state, regional and national levels and was a speaker at state and regional conferences. She has taught a college methodology course and a Peers Academy workshop at the local university for mathematics teaching. Sue has designed programs and taught workshops for in-service teachers across the state in conjunction with her local university. Callie conducts research and teaches mathematics knowledge and best practices to her peers within her school district. Kathy discussed the impact that influential people had in her professional development,

> I just think that over time reading, attending conventions, and listening to good math teachers and having the opportunity to hear someone that is enthusiastic about math and how they teach it in their classroom. That has had a lot of impact. It all comes down to opportunities and wanting those opportunities.

The teachers were also mathematical leaders in their school districts. Both Callie and Sue selected the textbooks that were used in all the elementary grades in their respective school districts. Callie piloted the textbook series and reported the findings to the district's administration for approval. Callie also helped develop a mathematics curriculum guide that determined for teachers what mathematics to teach and on what level of proficiency students should achieve in the third grade for her school district. Sue became the mathematics departmental teacher and taught mathematics to all students in the district from the fourth through sixth grades. The teachers have continued to progress professionally in their mathematical leadership in their school districts since the collection of data for this research study. Kathy has new roles in her school as "Math Intensive First Grade Teacher" and is the schools "Math Coach" for grades K-2. She has also completed a primary math program at the local university. Callie has moved into the "Math Facilitator" position of her school for all grade levels.

The teachers were committed to a life-long pursuit of mathematical growth and excellence. The Adding it Up report (2001) by the National Research Council states, "Teachers cannot automatically know how to teach more effectively. Learning to teach well cannot be accomplished once and for all in a preservice program; it is a career-long challenge" (p. 12). The teachers were involved with professional development activities because they realized these experiences helped them to become better teachers, helped them understand the mathematics better, and that their students would be the beneficiaries of the developed knowledge and skill being learned. The teachers searched for information and strategies that would enable all students from different backgrounds and levels of understanding to be successful and grow in their understanding of mathematics, just like the teachers in this study were able to accomplish in their own experiences. Kathy said,

> I'm a life-long learner. There is just something about the notion that you can know everything and just teach it one way and it's good. There's always a better way out there, you just have to find it. You have to keep searching and just because one way is a great way for one class doesn't mean you can use it the next year. It can be totally wrong for that group. There's always a better way, we're developing new ways all the time.

The teachers believed excellent mathematics teaching was an ongoing process that evolved throughout their professional lives rather than being a static, fixed procedure they could master. Sue said, "I have a passion for learning. I'm not one to sit and be idle." Callie also talked about her passion to develop as a mathematics teacher,

> I always want to do better. Always want to be better. There's always new research out there. There's always new strategies, new activities new ideas. I want to try them and I want to be better. Every year I have to improve as a teacher, otherwise, I'm not doing justice for the kids. Everything has to evolve to get better.

The excellent teachers in this study were progressive because they favored progress, change, and improvement, as opposed to wishing to maintain the teaching of mathematics they way it was and the way that they experienced it. They wanted to make progress toward more advantageous methods and continuous improvement so their students would enjoy and be more successful in learning mathematics. These teachers had a strong passion and an inner drive for the progressive excellence of teaching and learning mathematics. Each teacher pursued knowledge and skill in mathematics, on their own, that met their individual curiosity and the special needs of their students. They were not required to do any of these activities and sacrificed their personal time and efforts seeking more knowledge and skill that benefitted their students. Because these teachers were life-long seekers of mathematical knowledge and skill they were able to be the curriculum in their mathematics classrooms.

Chapter 5

Conclusions and Implications

This study asked leaders in mathematics and mathematics education to nominate elementary teachers they considered to be excellent. The top 3 of the 21 nominations submitted volunteered to participate in the study. These teachers took the CKTM instrument from the University of Michigan as a selection procedure to examine expertise of their knowledge and teaching of elementary mathematics. This instrument was implemented toward the end of each teacher's data collection process to eliminate potentially influencing teacher performance and became an aide for discussion. Each teacher was observed at least nine consecutive days during a unit of mathematics instruction during which three semi structured interviews were conducted. Field notes were collected and informal interviews were conducted with each teacher throughout the observations. Transcripts were developed for each teacher's interviews and observations. Codes were identified and winnowed into clusters of codes identifying similar ideas. Episodes were selected from each teacher's teaching that represented clusters, were everyday occurrences, or were unique experiences that contributed to the case forming a set of key issues for each teacher. The key issues were used in a cross-case analysis of all three teachers holistically creating five categories of correspondence or patterns. The categories were blended with research and direct interpretation into six dimensions for excellent teaching. A chart of this study's analyses process is included (see Figure 2).

The teachers were strikingly similar in their philosophies, teaching strategies, and beliefs about the teaching and learning of mathematics. These similarities

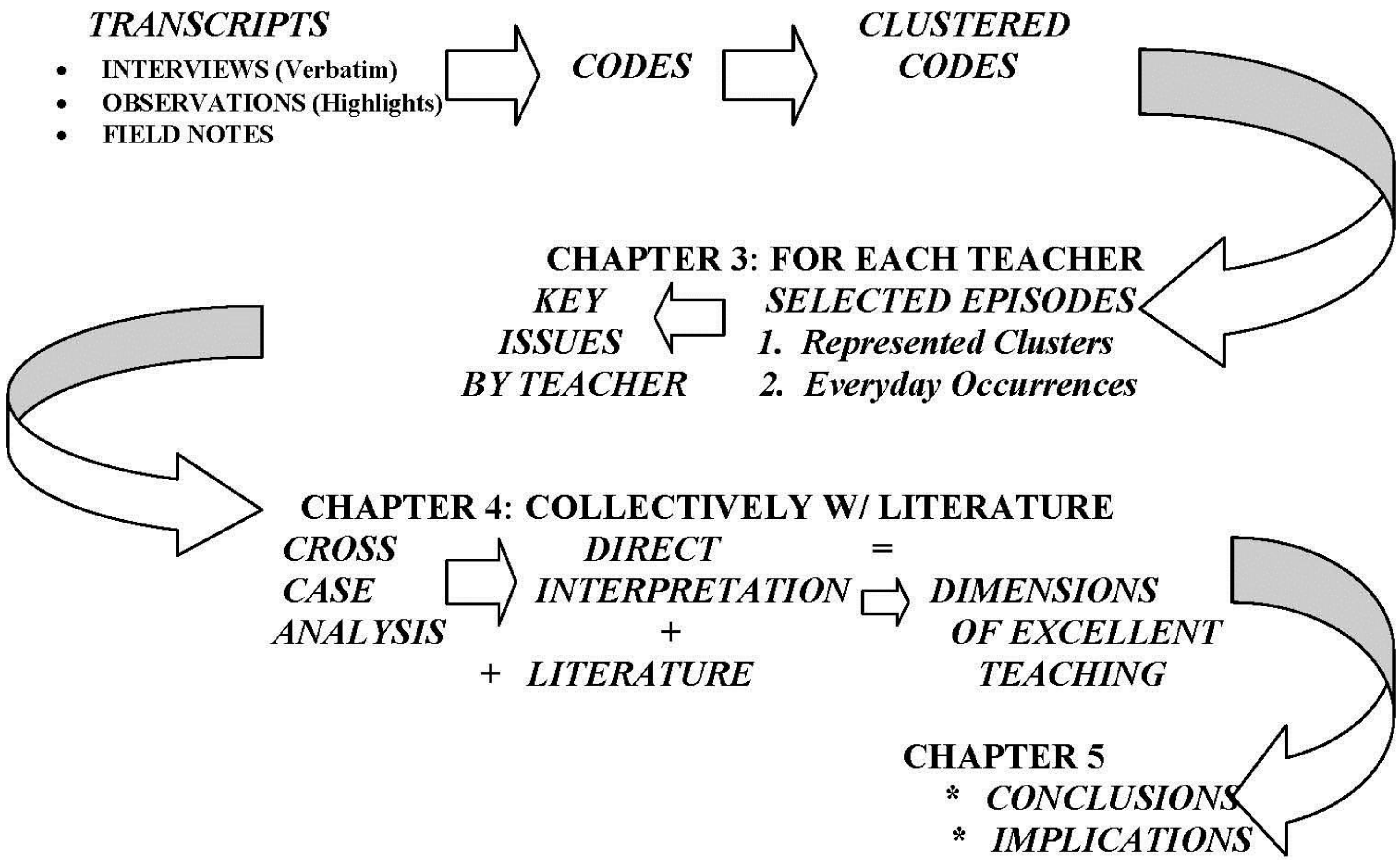

Figure 2. Analyses flowchart.

developed into eight conclusions that have been identified from this case study. The conclusions were based on the dimensions and are exemplars of what the excellent teachers in this study consider excellent elementary mathematics teaching to be. I also reconsidered the grand tour question of the study: How do three excellent elementary mathematics teachers in public schools in a Midwestern state teach mathematics in their classrooms? What did they know and what did they do?

Conclusions

Conclusion 1: Passion and determination for mathematics and students.

Excellent teaching was not determined to be different by grade level in this study. Grades one, three, and six were represented and the teachers' pedagogical decisions and student expectations and performance were the same on each of the levels. Excellent teaching was not determined by the context of the school setting. An inner city, suburban, and a rural community school were represented in the study. The teachers did not change their teaching strategies and had high expectations for students to be successful in all three settings. Mueller, Yankelewitz, & Maher (2011) found that, although the demographics of groups of students and the tasks may be different, the reasoning and subsequent understanding that occurs is quite similar. Excellent teaching was not determined by the teachers' mathematical education backgrounds. None of the teachers had extensive mathematical education, math degrees, or were mathematicians. Two of the teachers enjoyed their mathematical schooling with one loving it and the other teacher hating her mathematics schooling altogether. Two of the teachers stressed that they wished they would have taken more math classes in high school and college.

These teachers were excellent because of their determination and commitment to making a positive difference for their students' experiences in mathematics and for their passion for excellence and knowledge in the discipline. The teachers discovered in their teaching journeys they could understand the mathematics and wanted to teach their students what they had discovered. These teachers experienced a mentor or teacher along the way that inspired them to be excellent and showed them how they could effectively teach students mathematics differently then they experienced it. This inspiration sparked a life-long passion to seek more knowledge and skill to benefit and reach each of their students allowing them to overcome their fear of mathematics. They believed if they could learn and understand the mathematics, all of their students could learn it too.

Conclusion 2: Explanation/communication leads to understanding. This study indicated learning and understanding mathematics involved dialogue, interaction, and questioning skill. Students were very rarely sitting at their desks working on problems independently. The teachers were articulate and precise in their explanations and language in discussing the mathematics and expected their students to do the same. It was very important to them their students were able to articulate back correctly what they understood and learned. Students were assigned real world math problems to work on in pairs and small groups so they could verbally interact with others about the mathematics and learn from each other. Communication occurred between the teacher and students, students and teacher, and students to each other. Discussions occurred frequently on problems, strategies, discoveries, mathematics, and conjectures in groups, pairs and whole class formats. How the teachers explained tasks, ideas, student responses, and the mathematics was critical for students to understand the concepts and processes cognitively and metacognitively. The teachers modeled mathematical processes verbally and visually to help students understand how they could solve a problem mathematically.

These excellent teachers focused on how students were cognitively processing the mathematics in their brains allowing students to engage in mathematical reasoning. That is why the teachers were continuously seeking to understand what their students were thinking. How students were processing the mathematics conceptually determined how the teachers proceeded with the lesson. This was a key idea of the teachers' teaching because they were confident enough about their understanding and abilities of the teaching and learning of mathematics to become vulnerable by teaching spontaneously to students' explanations and interactions. Thinking and explaining that thinking helped the teacher and students understand the mathematics they were learning and created a zone of proximal development which became an avenue of cognitive growth for students. These teachers used specific questions strategically to probe for student understanding, guided students through processes, and developed deeper understandings from students' current level of understanding without telling them what to think or giving the answer.

Conclusion 3: Interwoven mixture of procedural and conceptual knowledge. This study searched for the answer to the controversial "math wars" (Daro, 2003; Schoenfeld, 2004) in determining what should be included in the excellent teaching of mathematics. Some believed teaching procedurally helped students learn needed algorithms and provided adequate practice to improve computational skill (Hechinger, 2006; Klein, 1999; Wu, 1997). Others believed teaching mathematics conceptually helped students understand what they were doing and why the mathematics worked

(Moses, 2001; Rosen, 2000; Stanic, 1987). It was thought these conceptual experiences allowed students to transfer skills to other settings. There seemed to be some disagreement from scholars fundamentally about what constituted excellent mathematics teaching: from a conceptual or procedural approach. This study and current research suggests there is an interdependent relationship between conceptual and procedural understandings and that both are needed for excellence. Although the teachers in this study taught their students to focus on understanding they frequently were teaching the understanding of both concepts and procedures. They interwove both procedural and conceptual aspects, strategically, of each of the mathematical ideas they taught so understanding the conceptual part of an idea contributed to the understanding of the procedural part of the idea and vice versa. They were continuously making connections between conceptual and procedural aspects of the mathematics. The teachers modeled and presented visual representations of these connections so their students developed deeper conceptual webs of domain knowledge and schemata of the discipline.

The teachers believed teaching concepts without procedures and vice versa was not effective teaching and they could not teach that way because their students would have incomplete partial understandings of the mathematics they were teaching. Even in Callie's classroom where there were no worksheets, homework and tests students were engaged in procedural tasks and understandings within the daily problem activities. For instance, in the ½ yellow and the 7 divided by 4 problems students needed to show procedural understanding of the concepts to solve the problems accurately and completely. The procedural learning in these classrooms were explored in the contexts of the problems presented which is fundamentally different from performing tasks on a series of problems on an unrelated and isolated worksheet. It allowed students to discover and develop their own level of understanding about the mathematical idea they were pursuing; therefore, making cognitive connections in each child's mental framework.

Conclusion 4: Focus on Understanding. The teachers in this study developed their own understandings of the mathematics they taught through experience, research, and professional development. They used this knowledge and skill to progress their students' understandings through problem solving activities, manipulating manipulatives and representations, group work, discussions, and asking questions. These strategic pedagogical experiences provided feedback to the teachers and students about what was understood, what was partially understood and what was not understood. Spontaneous reflection and reaction to that feedback allowed the teachers and students to interact with the mathematics at the level of understanding they were at currently. The teachers used this information to make instantaneous decisions about what questions to ask and tasks to implement so students could discover and construct newly developed mathematical understandings. This allowed students to progress their thinking by constructing new knowledge and understandings and identifying and modifying current misconceptions and incomplete understandings they currently held. Students focused on and constructed their own mathematical understandings and thinking by exploring their own conjectures and ideas through discoveries in experiential activities. This allowed the teachers to teach the mathematics the students needed to develop and helped them construct new and current understandings independently.

The major focus in these teachers' classrooms was on understanding the mathematics with less focus on how many responses a student correctly scored on a series of problems. They focused on the process of how students got the answer rather than on the answer itself. The teachers believed that if students understood the concepts and processes thoroughly they would be able to execute procedures successfully resulting in the correct answer. The teachers taught mathematics for understanding and put major emphasis on how students' were understanding the mathematics through verbal explanations, students' visual representations of the processes they used to solve a problem, and asking specific questions to probe student thinking and understanding of mathematical concepts and procedures.

Conclusion 5: Math taught in the context of interactive exploratory problems. The teachers asked their students to solve problems and taught mathematical concepts and procedures in the context of that problem by the way the students were interacting with the problem. The problems were usually real world problems students were interested in and could see the value of knowing and using the knowledge being taught. The problems were specific non-routine and open-ended problems that created opportunities for the students to ask questions of themselves, hypothesize, make generalizations, and create unique conjectures and algorithms to discuss and construct the mathematics. The teachers allowed students to interact with the problem without giving instructions about what concept or procedure to use. This strategy allowed students to use their prior knowledge about how to solve the problem and the underlying mathematics. This important information helped teachers understand what these students knew about the mathematics and what to focus on to develop those concepts and procedures. The teachers frequently extended these problems to check for transferability and execution of the students' understanding of the concepts and procedures in another context.

Interactive and exploratory problems also allowed students to pursue the mathematics in their own way and usually resulted in multiple pathways to solving the problems mathematically. The teachers encouraged students to solve problems in multiple ways and presented those discoveries to the class. Students were able to make personal contributions to the class about their invented algorithm, thinking processes, and discoveries that other students could learn from. The students were able to see other students' thinking and pathways to solving the same problem allowing all students to develop multiple strategies for solving problems. The teachers in this study used student work and thinking as examples to learn the concepts and procedures that were planned in the lesson. Students were able to examine each others' thinking and strategies and developed their own creativity and critical thinking skills. This showed students that there were many ways of finding solutions to problems which developed their metacognitive skills in logical sense making and fostered advanced mathematical thinking. These teachers taught their students to understand the mathematics flexibly and deeply so they could use the mathematical knowledge and skill in the context of the real world throughout their lives. The students built confidence and motivation for their ability to learn and use mathematics from these experiences because they were allowed to discover and use their own thinking and ideas to learn.

Conclusion 6: Externalization of mathematical processes and thinking. The teachers in this study externalized what was going on conceptually in their students' minds in a variety of ways. They asked specific questions to inquire and probe students' understanding of an idea, concept, or procedure. It was important to the teachers that they understood precisely what each student knew. They asked additional bundles of questions to help them identify the student's complete understanding. They were careful not to assume a student understood something without probing to confirm precisely what the student understood.

Another way the teachers externalized students' mathematical thinking and understanding was by asking students to work problems on individual boards or on the whiteboard so that they could see what processes the students were executing as they solved the problem. They insisted the students showed their work so that the teacher could analyze how they were executing the problem strategically. The teachers could see what the students were doing and could identify what they were thinking to execute a problem in that way and to find mistakes in the execution of a procedure or concept of the problem visually. The teacher and students were able to examine the thinking and processes of others and developed metacognitive knowledge and flexibility of problem solving strategies.

The teachers also externalized mathematical understandings and thinking through dialogue students were having with each other. The teachers organized students to work in small groups and pairs and listened to how the students were verbally explaining their thinking to each other. The teachers obtained valuable feedback about what the students knew and what they were thinking as they explained to each other their conceptual and procedural understandings. The externalized information informed teachers about how to proceed in the lesson to develop students' understanding. The teachers in this study made spontaneous decisions from students' external expressions about what questions to ask, what representation to present, when to correct, when to guide or model, when to let students grapple, who to call and in what order, and what to emphasize.

Students were able to benefit from externalizing the mathematics and gained autonomy. Their ideas were valued and used to learn and instruct others about personal mathematical concepts and procedures. The students felt that if the teacher used his or her thinking and discoveries as part of her teaching then it must be valuable and important. They also benefitted by seeing, discussing and examining other students' mathematical thinking. The students were able to compare their thinking to someone else's thinking and determine their own errors and strategies that work best for them personally.

Conclusion 7: Students as active agents of their own learning. The teachers expected all of their students to actively discover and construct their own mathematics learning. Students were allowed to be active agents in their learning because the teachers believed they were capable thinkers that could reason mathematically. Students were allowed to pursue and test their own conjectures and algorithms in attempting to solve problems and frequently would find multiple pathways to the same solution without being told what strategy to use. They were able to examine those findings with other students' ideas and presented their results to the class for analysis to determine if their ideas were proven and why. Students were expected to work in small groups, independently from the teacher, on problems through exploration and dialogue exposing and testing their thinking. They agreed or disagreed and provided reasoning for their decisions. Students taught each other using their own thinking and logic while the teacher facilitated and guided independent group learning through observations, listening, asking questions, and having students present their findings to the class for examination to clarify and explore students' conclusions and thinking. Students' work and thinking were valued and used in the teaching of mathematics. By valuing individual thinking, students were able to construct their own understandings. The teachers strategically looked for examples of students' work and thinking to present to the class for examination to develop specific aspects of the concepts and procedures of the mathematics they felt would advance students' understandings. Students were able to make contributions to the learning of the mathematics by presenting their findings, unique algorithms and strategies, and thinking to the class to examine and use to solve other problems. Teachers valued students' thinking by asking them to explain their thinking about the concepts and procedures they used.

The teachers in this study created a community of learners in their mathematics classrooms. The students were not as concerned about their ideas being correct or not because all of their thinking was valued and contributed to the learning and understanding of the mathematics. The focus was on the process and student thinking and less on the final product. Students were not penalized for their thinking or their incorrect answer so they felt comfortable sharing their ideas with the teacher and each other knowing the major goal of the class was for all students to understand the mathematics better. The students worked together on solving problems, making discoveries and communicating with each other, the teacher, and the class to help each student develop his or her own understanding. Each student's contribution, whether it be correct or not, advanced all the students learning. The teachers created a community of students who were active agents of their own learning that collectively advanced all students' mathematical understandings.

Conclusion 8: It's all about the students. Every decision that each of these teachers made were based on the specific needs of the students they were teaching. They continuously assessed what the students were saying and doing and asked questions to delve deeper into their thinking to learn what they knew and what they did not know. They used these techniques collectively with the class and individually with each student. The teachers taught students mathematics instead of teaching mathematics to students. The students were the focus of instruction for these teachers and every teaching decision was based on how students were understanding the mathematics through visual and verbal cues. The teachers were very diligent about knowing exactly what the students understood from those cues and usually would inquire deeper into their thinking to understand their understanding. This required the teachers to be spontaneous and teach the mathematics in the context of the students' experiences and understandings. The teachers in this study knew what they wanted to teach but did not know how they were going to get there until they were in the midst of the teaching and interacting with the students. They first wanted to learn where the students' current thinking and knowledge was with the mathematical idea and then guided the students from where they currently were to progress their understanding and skill to where the teachers wanted them to go. The teachers frequently diverted from their initial plans to meet the specific needs of their students.

The teachers were able to teach this way because they were the curriculum. Their passion and commitment to excellent teaching and students allowed them to acquire the knowledge and skill needed to teach in this way. The teachers were the curriculum because they had expertise in content knowledge, pedagogical content knowledge, and knowledge of students and mathematics. The teachers had a deep understanding of the big ideas of the discipline and the concepts, procedures and relationships of this content knowledge. The pedagogical knowledge these teachers possessed went beyond subject knowledge to the knowledge and skill of teaching the subject to students. The teachers knew what pedagogical strategies worked best for their students to construct and understand the mathematics they were teaching. The teachers had knowledge about how students learned mathematics that included logical sense making, indentifying misconceptions, errors in thinking and executing procedures, and examining their students' understandings to determine incomplete and incorrect conceptions. The teachers, throughout their careers, looked for professional development opportunities to develop their mathematical teaching skill and knowledge specifically for these three areas of expertise. They gradually learned what to do, how to do it, when to do what, why it worked, and multiple ways to execute it mathematically. The knowledge and skill that were developed in these teachers throughout their experience as professional educators allowed them to focus all of their teaching efforts on the students themselves. Teaching mathematics was contextual for these teachers and they taught mathematics by valuing individual and group differences instead of teaching mathematics by a one-size fits all philosophy.

Implications for Further Research

The findings of this study offer opportunities for further research in determining excellence in elementary mathematics teaching.

Recommendation 1. Further research is needed on how excellent teachers develop and use questioning strategies. It was apparent to me that the teachers in this study asked questions differently than other teachers. Studies should focus on the teacher's questioning strategies and how students respond to and use them. Research that looks at the sequence of questions a teacher uses and the reasons behind those choices would help scholars determine the kinds of questions and in what sequence for specific reasons can be developed in pre-service and in-service teacher skill to student outcomes (Franke et al., 2007). These studies could identify what kinds of questions excellent teachers ask and why they asked those questions. An advanced method of determining the kinds of questions a teacher asks also needs to be developed. Research also needs to occur on what teachers say in dialogue and explanation with students before and after a question is asked and how that affects how students respond and understand the question or concept that is being probed and how these interactions affect student cognition. Other questions that researchers can explore in this kind of study include: Are questions prepared in advance of a lesson and in what way? If so, how are they used and modified in the midst of the lesson? What are the purposes of the questions a teacher asks? What kinds of questions do students ask in response? The study of excellent teacher questioning strategies can help teachers develop and improve their own questioning patterns and how questions affect learning in the classroom for students.

Recommendation 2. Longitudinal studies are needed to follow students' continued math education experiences from elementary throughout high school and how this type of teaching affected their performance, what was retained and transferrable, how they solved problems throughout their education, and what were their efficacy beliefs.

This would inform scholars and researchers of the impact specific teaching strategies had on advanced mathematical study. Longitudinal research could also be conducted on teachers' mathematical teaching journey. There is a need to understand how a teacher progresses from a novice teacher to an expert teacher. Examining the journey could identify what progresses and hinders growth and how these experiences affected the teachers' teaching, their philosophy, and what they ended up teaching. The teachers in this study did not start their careers as excellent mathematics teachers or necessarily planned to teach the subject. The journeys of the teachers in this study were extraordinary and could contribute to how a teacher develops to an excellent level. This knowledge could contribute to professional development programs to help all teachers achieve excellence.

This study focused on what excellent teachers knew and what they did in their teaching. Studies that focused on the learner's experiences could inform research on the relationships and effectiveness of the pedagogical decisions of the teachers. How are teachers' and students' experiences interwoven and interdependent?

Recommendation 3. Larger meta-analysis and comparative studies could be conducted to examine potential patterns of excellent mathematics teaching with national and/or international perspectives for stronger generalizability. This study researched the teaching of three strikingly similar educators with virtually identical philosophies and strategies. Although this relationship provided important information about teaching excellence it leaves questions about what excellent elementary mathematics teaching is for teachers of varying philosophies and strategies of teaching mathematics. Research is needed for comparing differences in excellence based on different teaching philosophies and teaching strategies. This could allow scholars to develop more informed definitions of what excellent teaching is and explore the relationships between differing philosophies.

Recommendation 4. Additional research is needed on the effects and dynamics of social and group interactions and how these interactions affect student learning. In what ways, if any, does this type of teaching help students understand mathematics. The teachers in this study used dialogue, communication and groups allowing students to converse about the mathematics. Studies that examine deeply how students interpret and perform based on those experiences. Studies that focus on students' social and group discourse could inform practitioners about what helps students' cognition and what hinders it. It could help teachers inform students about how to interact in groups and how to metacognitively analyze discourse. It could also inform teachers on how to monitor and facilitate these social interactions.

Recommendation 5. Additional research is needed in how excellent elementary teachers use curricular materials and make planning decisions about how they choose to teach. This study did not study what curriculum was used and the decisions these excellent teachers made to the written and intended uses in their enacted teaching. Studying this phenomenon could enhance our understanding more deeply about what constitutes excellent mathematics teaching.

Implications for Teacher Education

When looking at implications for teacher education programs it is important to look at two levels of development for teachers: pre-service and in-service. Teacher development professionals should make distinctions between the two levels of participants when developing programs. In-service teachers have experiences that contribute to the development of teaching practice that pre-service teachers usually do not have. There are three implications from this study for the development of teacher education programs. The same implications can and should be developed for each level of the participants' experience.

Recommendation 1. It is clear from this study that teacher education programs should teach mathematical instruction interdependently and intertwined. Pedagogical content knowledge should be taught in the context of content knowledge and at the same time with emphasis on how students learn that knowledge or skill. Procedural and conceptual understandings should be interwoven into teacher strategies, mathematics, and how students learn. This should be combined with how students represent and solve problems and how students explain and reason about the mathematics. The teachers in this study believed professional development experiences that emphasized how teachers taught and how students understood the mathematics were beneficial and preferred over other types of programs. This indicated programs should be developed that express practical teaching and learning experiences that identify the multiple complexities of the discipline and focus on teacher and student understandings. This deeply complex knowledge and skill can be obtained and used in teachers' teaching with sustained growth over time as shown by the teachers of this study. Programs should avoid, as much as possible, isolated learning experiences that concentrate on one specific aspect of the teaching or learning of mathematics.

Recommendation 2. Teacher education programs should design instruction to maximize social interaction and communication. Small groups or pairs should be assembled to explore and investigate the mathematics together in a supportive and non-judgmental way. These explorations should focus on both the teacher and the students' perspective and attempt to figure out how each thinks and understands the content. Whole class discussions should be implemented to engage authentic exploration of teachers and students representations of perceptions of the mathematical concepts and procedures. Participants should develop questioning strategies by developing different kinds of questions they could ask students in specific contextual situations. Practical experiences could be implemented where teachers could learn pedagogical knowledge, content knowledge and student knowledge in the context of authentic teaching. Participants should have opportunities to analyze teaching episodes together to understand what was happening and why. This could help teachers understand pedagogical strategies to content that is specific to student understanding and learning of the material. Participants should examine how teachers articulate explanations and explore effective ways to respond to student contributions and dialogue. They should also investigate ways to create problems and activities, multiple solution pathways, and how and what manipulatives you can use.

One practical model of how a program could be designed as implicated above is through situational teaching episodes where the participants and their exploration and discoveries are the focus of the program. The program could be designed using important mathematical ideas. Each small group of two or three participants would conduct research on their big idea and identify their deep understandings and connections. They would create problems or activities to teach procedural and conceptual understandings of their big idea interdependently identifying aspects of the mathematics that students may have difficulty understanding or executing. They would then develop a series of questions they may ask generally and specifically for possible situations that may occur during instruction. They could identify and understand multiple ways students may strategically attempt to solve the problem or activity. Manipulatives or other experiences could be developed by the group to allow students to actively construct their own knowledge and understandings. Pre and post classroom discussions could be used before the groups actually teach their big idea lesson to actual students where their plan could be put into practice. These discussions could allow for refinements and possible ideas and understandings the group may not have thought about from other teachers and colleagues. The teaching could be video taped for the class to analyze and explore the development of teaching and learning experiences of mathematics in all of its complexities in authentic practice. This could provide participants a model of excellent teaching they could use in their teaching journeys as they develop deeper personal understandings of mathematical concepts and procedures they will be teaching. This model can be used in any of the mathematical big ideas participants may teach in the future and deepens their mathematical content knowledge in the context of pedagogical skills and knowledge of students and mathematics. Investigations can explore the relationships between each group's big idea to grasp deeper understandings of mathematical ideas and how they are interdependent and intertwined. This could provide an environment where instructors can facilitate and guide participants' construction of understandings of the teaching and learning of relevant mathematics in a collaborative social interactive setting.

Recommendation 3. This study suggests that teacher education programs should emphasize that excellence is a life-long exploration of developing knowledge and skill in the teaching and learning of mathematics. Teacher education programs should stress advanced experiences in the teaching and learning of mathematics and should help students develop a plan for their professional development career by showing the benefits of quality experiences and identifying professional development opportunities that are available.

The excellent teachers in this study believed that mathematics is not static and new and improved strategies and understandings are being developed continuously to aid in helping teachers benefit their students' understandings and skills of what they teach and want to teach in the future. This study showed that research and professional development has allowed these teachers to enhance their students' mathematical experiences and be successful in learning and understanding mathematics. They understood the mathematics deeply and developed multiple pathways to meet the specific needs of their students.

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Appendix A

IRB Approval

University of Nebraska Lincoln

HUMAN RESEARCH PROTECTIONS
Institutional Review Board

February 4, 2008

Michael Gay
Dr. Larry Walter
6000 Meridian Dr. #10
Lincoln, NE 68504

IRB# 2008-02-8591 EX

TITLE OF PROJECT: **Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics**

Dear Michael:

This letter is to officially notify you of the conditional approval of your project by the Institutional Review Board (IRB) for the Protection of Human Subjects. This project has been approved by the Unit Review Committee from your college and sent to the IRB. It is the Board's opinion that you have provided adequate safeguards for the rights and welfare of the participants in this study. Your proposal seems to be in compliance with this institution's Federal Wide Assurance 00002258 and the DHHS Regulations for the Protection of Human Subjects (45 CFR 46) and has been classified as exempt.

Date of EX Review: **01/17/08**

You are authorized to implement this study as of the Date of Final Approval: **02/04/08.** This approval is Valid Until: **02/03/09.**

1. Uploaded on NUgrant is the IRB approved Informed Consent form for this project. Please use this form when making copies to distribute to your participants. If it is necessary to create a new informed consent form, please send us your original so that we may approve and stamp it before it is distributed to participants.

2. Your project has been given conditional approval. You will receive final approval after you have submitted an approval letter from one of the schools you are working with. Additional letters can be submitted on a case by case basis as you receive them. Please email me the school approval letters. Now that your project has been approved, you will not be able to upload documents to your protocol.

3. Please submit the questions for the 2nd and 3rd interviews as a change in protocol.

We wish to remind you that the principal investigator is responsible for reporting to this Board any of the following events within 48 hours of the event:

- Any serious event (including on-site and off-site adverse events, injuries, side effects, deaths, or other problems) which in the opinion of the local investigator was unanticipated, involved risk to subjects or others, and was possibly related to the research procedures;
- Any serious accidental or unintentional change to the IRB-approved protocol that involves risk or has the potential to recur;
- Any publication in the literature, safety monitoring report, interim result or other finding that indicates an unexpected change to the risk/benefit ratio of the research;
- Any breach in confidentiality or compromise in data privacy related to the subject or others; or
- Any complaint of a subject that indicates an unanticipated risk or that cannot be resolved by the research staff.

This project should be conducted in full accordance with all applicable sections of the IRB Guidelines and you should notify the IRB immediately of any proposed changes that may affect the exempt status of your research project. You should report any unanticipated problems involving risks to the participants or others to the Board. For projects which continue beyond one year from the starting date, the IRB will request continuing review and update of the research project. Your study will be due for continuing review as indicated above. The investigator must also advise the Board when this study is finished or discontinued by completing the enclosed Protocol Final Report form and returning it to the Institutional Review Board.

209 Alexander Building West / 312 N. 14th Street / P.O. Box 880408 / Lincoln, NE 68588-0408 / (402) 472-6965 / FAX (402) 472-6048

UNIVERSITY OF Nebraska Lincoln

HUMAN RESEARCH PROTECTIONS
Institutional Review Board

If you have any questions, please contact Shirley Horstman, IRB Administrator, at 472-9417 or email at shorstman1@unl.edu.

Sincerely,

Dan R. Hoyt, Chair
for the IRB

209 Alexander Building West / 312 N. 14th Street / P.O. Box 880408 / Lincoln, NE 68588-0408 / (402) 472-6965 / FAX (402) 472-6048

University of Nebraska Lincoln

January 5, 2009

Michael Gay
Teaching, Learning and Teacher Education
505 Court St. Beatrice, NE 68310

Larry Walter
Teaching, Learning and Teacher Education
238 MABL UNL 68588-0234

IRB Number: 2008-02-8591
Project ID: 8591
Project Title: Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics

Dear Michael:

This is to officially notify you of the approval of your project's Continuing Review by the Institutional Review Board for the Protection of Human Subjects. It is the committee's opinion that you have provided adequate safeguards for the rights and welfare of the subjects in this study based on the information provided. Your proposal is in compliance with DHHS Regulations for the Protection of Human Subjects (45 CFR 46).

We wish to remind you that the principal investigator is responsible for reporting to this Board any of the following events within 48 hours of the event:
• Any serious event (including on-site and off-site adverse events, injuries, side effects, deaths, or other problems) which in the opinion of the local investigator was unanticipated, involved risk to subjects or others, and was possibly related to the research procedures;
• Any serious accidental or unintentional change to the IRB-approved protocol that involves risk or has the potential to recur;
• Any publication in the literature, safety monitoring report, interim result or other finding that indicates an unexpected change to the risk/benefit ratio of the research;
• Any breach in confidentiality or compromise in data privacy related to the subject or others; or
• Any complaint of a subject that indicates an unanticipated risk or that cannot be resolved by the research staff.

It is the responsibility of the principal investigator to provide the Board with a review and update of the research project each year the project is in effect. This approval is valid until 02/02/2010.

If you have any questions, please contact the IRB office at 472-6965.

Sincerely,

Mario Scalora, Ph.D.
Chair for the IRB

Appendix B

Letter of Nomination

UNIVERSITY OF Nebraska Lincoln

COLLEGE OF EDUCATION AND HUMAN SCIENCES
Department of Teaching, Learning and Teacher Education

Letter for Recommendations

Deborah Romanek
Mathematics Education Director
Nebraska Department of Education
301 Centennial Mall South
P.O. Box 94987
Lincoln, NE 68509

Dear Dr. Romanek:

My name is Michael Gay, Ph.D. candidate, from the University of Nebraska-Lincoln. I am conducting my dissertation research in elementary mathematics. The goal of my research is to describe and understand what excellent elementary mathematics teaching is and looks like in the classroom. The title of my study is: Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics.

I am searching for nominations for excellent elementary mathematics teachers and hope that you could provide me with a few recommendations as possible teachers to study in my dissertation. I am looking for three of the very best elementary mathematics teachers in Nebraska and the surrounding states. You represent one of a number of educational organizations I am contacting for nominations!

The requirements for nominations include:

1. The nominee must be a practicing teacher in an elementary school setting.
2. The nominee must currently be teaching mathematics and is selected for his or her teaching of mathematics.

Please fill out your nominations on the attached nomination form, sign the voluntary consent form and return them in the self-addressed envelope. Your help in finding the best elementary mathematics teachers to voluntarily participate in this study is truly appreciated. You and your nominations will be kept anonymous. I will be the only person that will see your recommendations and your confidentiality will be strictly kept. I may want to call you to obtain more information about your selection of a candidate. Nominations will not be contacted unless they make the final list of candidates and are considered to be the very best candidates.

118 Henzlik Hall / P.O. Box 880355 / Lincoln, NE 68588-0355 / (402) 472-2231 / FAX (402) 472-2837

Please contact me any time if there are any questions that you may have regarding this research study. Your participation and efforts in this selection process is greatly appreciated.
Sincerely,

Michael Gay, University of Nebraska-Lincoln
238 Mabel Lee Hall (J.Walter)
P.O. Box 880234
Lincoln, NE 68588-0234
(402)217-0672

Directions for the Nomination Form:

1. Complete a section for each candidate that you nominate. You do not have to complete all three sections if you are not nominating three candidates. You may also add more candidates on the back or on additional paper if you are nominating more than three candidates. Include the candidates name, school, city of school, and a rationale for nomination.
2. You may also use additional paper or the back of the form if there is not enough room to complete your writing.
3. Rationale for nomination section is for you to tell why you are selecting this candidate as an excellent elementary mathematics teacher.

Excellent Teachers: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics

Nomination Form

Teachers Name: ______________________

Teacher's School: ______________________ City: ________________

Rationale for Nomination:

Teacher's Name: ____________________________

Teacher's School: ___________________________ City: _____________________

Rationale for Nomination:

Teacher's Name: ____________________________

Teacher's School: _____________________________ City: ______________________

Rationale for Nomination:

Appendix C

Informed Consent Form

COLLEGE OF EDUCATION AND HUMAN SCIENCES
Department of Teaching, Learning, and Teacher Education

Nebraska UNIVERSITY OF
Lincoln

Informed Consent Form

IRB# 2008028591 EX
Date Approved: 02/04/2008
Valid Until: 02/03/2009

Identification of Project:
Title: Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics

Purpose of the Research:
This is a research project of purposely selected excellent elementary mathematics teachers to describe and understand what excellent elementary mathematics teaching looks like in the classroom. By learning about excellent elementary mathematics teaching we can develop better training programs for pre-service teachers and help practicing teachers through better professional development programs. Teachers for this study come from recommendations by administrators and educational organizations as outstanding mathematics teachers that positively affect student achievement.

Procedures:
This study asks teachers to become co-researchers with the researcher. This study includes multiple classroom observations of the teaching of mathematics for approximately 2 weeks (during a unit or chapter), three formal interviews of approximately one hour each, and a survey taken by the co-researcher. All of these activities will be conducted in the co-researcher's classroom at the co-researcher's convenience. The observations will take place at a time that has been set up by the co-researcher and researcher. The researcher will sit in the classroom and take notes while the teacher teaches lessons during a complete unit, chapter, or concept. The interviews will be face to face and take approximately one hour. The interviews will be audio-taped for the researcher to record information the co-researcher gives and will be
will be completed by the co-researcher in private at the co-researcher's
ice questions about teaching mathematics in classrooms and identifies your
t will take about one hour of the co-researcher's time and the focus of the survey
the co-researcher on how excellent teachers think about the questions. All of the data collected will only be seen by the researcher and the co-researcher. The researcher will discuss all of the data with the co-researcher throughout the study and the co-researcher can ask questions at any time. The research will be conducted during the spring semester of the 2007-2008 school year.

Confidentiality:
The co-researcher will receive a pseudonym so that his or her identity remains confidential. Schools will not be identified in the study. Any information obtained during this study which could identify you will be kept strictly confidential. The data will be stored in a locked file cabinet in the researcher's office and will only be seen by the researcher during the study and for three years after the study is complete. The information in this study may be published in scientific journals or presented at scientific meetings but the data will be reported keeping you and your school's identity strictly confidential.

Opportunity to Ask Questions:
You may ask any questions concerning this research study and have those questions answered before agreeing to participate in or during the study. You may call the researcher at any time at 402-217-0672. You can also contact the researcher via e-mail at mgay1@neb.rr.com. If you have questions concerning your rights as a co-researcher that have not been answered by the researcher, or to report any concerns about the study, you may contact the University of Nebraska-Lincoln Institutional Review Board at (402) 472-6965.

IRB# 2008028591 EX
Date Approved: 02/04/2008
Valid Until: 02/03/2009

Risks/Discomforts:
There are no known risks or discomforts associated with this research.

Benefits:
The information gained from this research may help us to better understand what excellent elementary mathematics is and looks like. These discoveries can assist us in improving mathematics training for pre-service and practicing teacher programs. Learning about the co-researcher's own teaching of mathematics can benefit the co-researcher by identifying teaching strengths.

Freedom to Withdraw:
You are free to decide not to participate in this study or to withdraw at any time without adversely affecting your relationship with the researcher, the University of Nebraska-Lincoln, or your school.

Consent, Right to Receive a Copy:
You are voluntarily making a decision whether or not to participate in this research study. Your signature certifies that you have decided to participate having read and understood the information presented. You will be given a copy of this consent form to keep.

✓ Check if you agree to be audio taped

Signature of Co-Researcher (teacher):

Date: 4-10-08

Contact Information:
Mike Gay, M.A., Principal Researcher,
6000 Meridian Dr. # 10
Lincoln, NE 68504
Mgay1@neb.rr.com
(402) 217-0672

Jim Walter, Ed. D.
Associate Dean, College of Education and Human Sciences
238 Mabel Lee Hall
P.O. Box 880234
Lincoln, NE 68588-0234
(402) 472-3392

Appendix D

Protocols

Interview Protocol # 1

Interviewee: ______________________________ Date: ____________________

Pseudonym: _________________________ Location: ____________________

Interviewer: _________________________

Questions for interview protocols #2 & #3 will be submitted to IRB as a change in protocol once they have been developed.

Introduction:

I want to thank you for taking the time to be interviewed today. What we discuss will be recorded and later transcribed. I will be asking you to review the transcripts. It is important that I reflect in my writing what you mean. You will have the opportunity to verify, add and comment on what was experienced.

In this Interview, I am interested in getting to know you, your experiences, beliefs, training, and experience as a teacher. I want to know your perspective, so feel free to discuss your views. I may ask you some additional questions in order to clarify for me what you mean. Are you ready to start?

---

1. Briefly describe yourself. Notes:

    Probes:
    Hobbies
    Personal- family, kids, etc.

2. Tell me about your professional training and certifications.

    Probes:
    Advanced degrees (MA/PhD, etc.)
    Special training experiences
    Quality and preparedness of experiences

3. Tell me about your teaching experience.

    Probes:
    Years taught, schools taught at, grade levels
    Professional organizations
    Professional developments or programs

4. Tell me about your decision to become a teacher.

5. Tell me about your mathematics background.
   Probes:
   Elementary, Middle, High, College experiences
   w/parents
   examples

6. How do you feel about mathematics?
   Probe:
   Has it always been that way?
   Students feel about it?

7. What is mathematics to you?
   Probe:
   What are your strengths?
   What are your weaknesses?

8. Tell me about one of your favorite math lessons.
   Probes:
   How did the students react?
   What was it like for you?

9. Tell me about one of your worst math lessons?
   Probe:
   How did the students react?
   What was it like for you?

10. How do students learn mathematics?
    Probe:
    Example

11. What do you want your students to know and do in mathematics?
    Probe:
    Example

12. How do you teach mathematics?
    Probe:
    What is your role?
    What is the role of students?
    Give an example.

13. What does an elementary teacher need to know and do to teach mathematics well?
    Probe:
    example

14. How have parents responded to your teaching of mathematics?
    Probe:
    Example.

15. What else can you tell me about yourself, teaching, or mathematics that we haven't discussed yet?

Interview Protocol #2
Teacher # 1

Interviewee: ______________________________ Date: _________________

Pseudonym: ______________________________ Location: _________________

Interviewer: ______________________________

Introduction:

I want to thank you for taking the time to be interviewed today. What we discuss will be recorded and later transcribed. I will be asking you to review the transcripts. It is important that I reflect in my writing what you mean. You will have the opportunity to verify, add and comment on what was experienced.

In this Interview, I am interested in learning about your journey as a mathematics teacher, your lesson and unit development strategies, and your assessment practices and philosophy. I want to know your perspective, so feel free to discuss your views. I may ask you some additional questions in order to clarify for me what you mean. Are you ready?

---

1. How do you analyze student understandings of a lesson?
   Probes:
   How do you divert from a lesson plan? How?
   Symmetry to equality example

2. How do you decide the pace of your lesson?
   Probe:
   Unit?

3. How do you determine the sequencing of activities in a lesson?

4. How do you assess and grade your students in mathematics?
   Probes:
   Individually
   Identify strengths & weaknesses

5. How do your students do on standardized assessments?
   Probe:
   Students come back later

6. How did you get from hating to loving and being a leader in mathematics teaching?
   Probes:
   Examples of a lesson at beginning/ending
   "Evolving"
   Attitude
   Time spent on subject
   Epiphanies

7. How do you develop your questioning strategies?
   Probes:
   What are you thinking?

8. What motivates you to continue to develop your teaching skills in mathematics?

9. How do you select manipulatives, representations, and examples for your mathematics lessons?

10. What process do you go through when you develop a mathematics lesson?
    Probe:
    How long?
    What is your thinking?
    How do you put a unit together?

11. How do determine what you are going to do next?
    Probe:
    How long to stay on a concept?
    Is this the same every year?

12. What else can you tell me about your mathematics teaching journey that we have not discussed yet?

Interview Protocol # 3
Teacher # 2

Interviewee: ____________________________ Date: _______________

Pseudonym: ___________________________ Location: ________________

Interviewer: ___________________________

Introductions:
I want to thank you for taking the time to be interviewed today. What we discuss will be recorded and later transcribed. I will be asking you to review the transcripts. It is important that I reflect in my writing what you mean. You will have the opportunity to verify, add, and comment on what was experienced.

In this interview, I am interested in learning more about your teaching methods and strategies, and we will discuss the work of teaching mathematics with the survey. I want to know your perspectives and ideas, so feel free to discuss your views. I may ask you some additional questions in order to clarify for me what you mean. Are you ready?

---

1. How important is feedback in mathematics teaching?
Probe: today's lesson

2. How do you reflect on your teaching of mathematics?
Probe: Example

3. How do you modify a lesson from the text?
What other resources do you use for developing lessons?
Example

4. How has your mathematics teaching changed/evolved since you started eight years ago?

5. How do you modify a lesson for struggling math students?

6. How do teachers develop their mathematic teaching?

7. How do you teach your students to work in groups?
Math only subject?

8. How do you analyze a WS for use in your class?
Process
Thinking

9. What else can you tell me about your mathematics teaching journey that we have not discussed yet?

Discussion about the survey

1. What did you think about the survey? How did it make you feel?

2. Go over the results – Feel free to discuss what ever comes to your mind as we go.

Observation Protocol # 1

Location: __________________________ Time: ________________

Date: __________________ Length of Observation: ______________

Observer: _____________________ Lesson/Event: ______________________

Participants will be identified as student, teacher, etc. on protocol and not identifiers, such as the specific name of the participant.

Description of Event:

Participants:

Description of setting:

Physical map of the setting:

Observation # 1
Page # 2

Descriptive Notes	Reflective Notes
What is Happening?	
Who's in the group? (Students/Teachers, etc.)	
Interactions among participants	
Behaviors of participants	

Observation # 1
Page # 3

Types of Instruction/ Participant Roles	
Individual/ Group Operations	

Verbal Language	Body Language	Activities/ Materials

Appendix E

Released Survey Items

STUDY OF INSTRUCTIONAL IMPROVEMENT/
LEARNING MATHEMATICS FOR TEACHING

CONTENT KNOWLEDGE FOR TEACHING MATHEMATICS MEASURES (CKT-M MEASURES)

MATHEMATICS RELEASED ITEMS
2005

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 Measures development supported by NSF grants REC-9979873, REC- 0207649, EHR-0233456 & EHR 0335411, and by a subcontract to CPRE on Department of Education (DOE), Office of Educational Research and Improvement (OERI) award #R308A960003.

Study of Instructional Improvement/Learning Mathematics for Teaching

Content Knowledge for Teaching Mathematics Measures (CKT-M measures)
Released Items, 2005
ELEMENTARY CONTENT KNOWLEDGE ITEMS

1. Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true? (Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

2. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
x 25	x25	x 25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

3. Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.

4. Ms. Chambreaux's students are working on the following problem:

Is 371 a prime number?

As she walks around the room looking at their papers, she sees many different ways to solve this problem. Which solution method is correct? (Mark ONE answer.)

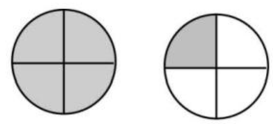
a) Check to see whether 371 is divisible by 2, 3, 4, 5, 6, 7, 8, or 9.

b) Break 371 into 3 and 71; they are both prime, so 371 must also be prime.

c) Check to see whether 371 is divisible by any prime number less than 20.

d) Break 371 into 37 and 1; they are both prime, so 371 must also be prime.

5. Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)



a) 5/4

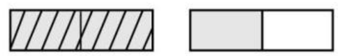
b) 5/3

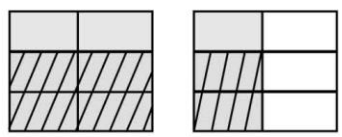
c) 5/8

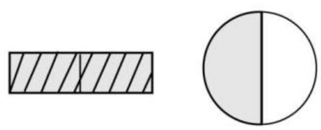
d) 1/4

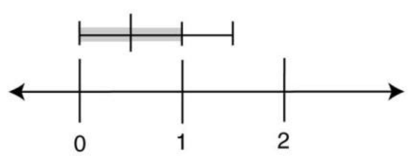
6. At a professional development workshop, teachers were learning about different ways to represent multiplication of fractions problems. The leader also helped them to become aware of examples that do <u>not</u> represent multiplication of fractions appropriately.

Which model below <u>cannot</u> be used to show that $1\frac{1}{2} \times \frac{2}{3} = 1$? (Mark ONE answer.)

A) 

B) 

C) 

D) 

7. Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have $1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2	3

8. As Mr. Callahan was reviewing his students' work from the day's lesson on multiplication, he noticed that Todd had invented an algorithm that was different from the one taught in class. Todd's work looked like this:

$$\begin{array}{r} 983 \\ \underline{\times\ \ 6} \\ 488 \\ +\underline{5410} \\ 5898 \end{array}$$

What is Todd doing here? (Mark ONE answer.)

a) Todd is regrouping ("carrying") tens and ones, but his work does not record the regrouping.

b) Todd is using the traditional multiplication algorithm but working from left to right.

c) Todd has developed a method for keeping track of place value in the answer that is different from the conventional algorithm.

d) Todd is not doing anything systematic. He just got lucky – what he has done here will not work in most cases.

ELEMENTARY KNOWLEDGE OF STUDENTS AND CONTENT ITEMS

9. Mr. Garrett's students were working on strategies for finding the answers to multiplication problems. Which of the following strategies would you expect to see some elementary school students using to find the answer to 8 x 8? (Mark YES, NO, or I'M NOT SURE for each strategy.)

	Yes	No	I'm not sure
a) They might multiply 8 x 4 = 32 and then double that by doing 32 x 2 = 64.	1	2	3
b) They might multiply 10 x 10 = 100 and then subtract 36 to get 64.	1	2	3
c) They might multiply 8 x 10 = 80 and then subtract 8 x 2 from 80: 80 – 16 = 64.	1	2	3
d) They might multiply 8 x 5 = 40 and then count up by 8's: 48, 56, 64.	1	2	3

10. Students in Mr. Hayes' class have been working on putting decimals in order. Three students — Andy, Clara, and Keisha — presented 1.1, 12, 48, 102, 31.3, .676 as decimals ordered from least to greatest. What error are these students making? (Mark ONE answer.)

a) They are ignoring place value.

b) They are ignoring the decimal point.

c) They are guessing.

d) They have forgotten their numbers between 0 and 1.

e) They are making all of the above errors.

11. You are working individually with Bonny, and you ask her to count out 23 checkers, which she does successfully. You then ask her to show you how many checkers are represented by the 3 in 23, and she counts out 3 checkers. Then you ask her to show you how many checkers are represented by the 2 in 23, and she counts out 2 checkers. What problem is Bonny having here? (Mark ONE answer.)

a) Bonny doesn't know how large 23 is.

b) Bonny thinks that 2 and 20 are the same.

c) Bonny doesn't understand the meaning of the places in the numeral 23.

d) All of the above.

12. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

$$
\text{I)}\quad \begin{array}{r} {}^{1} \\ 38 \\ 49 \\ +\ 65 \\ \hline 142 \end{array}
\qquad\qquad
\text{II)}\quad \begin{array}{r} {}^{1} \\ 45 \\ 37 \\ +\ 29 \\ \hline 101 \end{array}
\qquad\qquad
\text{III)}\quad \begin{array}{r} {}^{1} \\ 32 \\ 14 \\ +\ 19 \\ \hline 64 \end{array}
$$

Which have the same kind of error? (Mark ONE answer.)

a) I and II

b) I and III

c) II and III

d) I, II, and III

13. Ms. Walker's class was working on finding patterns on the 100's chart. A student, LaShantee, noticed an interesting pattern. She said that if you draw a plus sign like the one shown below, the sum of the numbers in the vertical line of the plus sign equals the sum of the numbers in the horizontal line of the plus sign (i.e., 22 + 32 + 42 = 31 + 32 + 33). Which of the following student explanations shows sufficient understanding of <u>why this is true</u> for all similar plus signs? (Mark YES, NO or I'M NOT SURE for each one.)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

	Yes	No	I'm not sure
a) The average of the three vertical numbers equals the average of the three horizontal numbers.	1	2	3
b) Both pieces of the plus sign add up to 96.	1	2	3
c) No matter where the plus sign is, both pieces of the plus sign add up to three times the middle number.	1	2	3
d) The vertical numbers are 10 less and 10 more than the middle number.	1	2	3

14. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students around particular difficulties that they are having with subtracting from large whole numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

$$
\begin{array}{ccccc}
\text{I} & \qquad & \text{II} & \qquad & \text{III} \\[1em]
\begin{array}{r}
\overset{4}{\cancel{5}}0\overset{12}{\cancel{2}} \\
-\ \ 6 \\ \hline
406
\end{array}
& &
\begin{array}{r}
3\overset{4}{\cancel{5}}00\overset{15}{\cancel{5}} \\
-\quad\ 6 \\ \hline
34009
\end{array}
& &
\begin{array}{r}
\overset{6}{\cancel{7}}\overset{9}{\cancel{0}}\overset{8}{\cancel{0}}\overset{15}{\cancel{5}} \\
-\ \ 7 \\ \hline
6988
\end{array}
\end{array}
$$

Which have the same kind of error? (Mark ONE answer.)

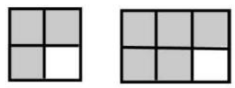
a) I and II

b) I and III

c) II and III

d) I, II, and III

15. Takeem's teacher asks him to make a drawing to compare $\frac{3}{4}$ and $\frac{5}{6}$. He draws the following:



and claims that $\frac{3}{4}$ and $\frac{5}{6}$ are the same amount. What is the <u>most likely</u> explanation for Takeem's answer? (Mark ONE answer.)

a) Takeem is noticing that each figure leaves one square unshaded.

b) Takeem has not yet learned the procedure for finding common denominators.

c) Takeem is adding 2 to both the numerator and denominator of $\frac{3}{4}$, and he sees that that equals $\frac{5}{6}$.

d) All of the above are equally likely.

Appendix F

Categorical Aggregation Chart

Karen
Day #1 10-5-09
Categorical Aggregation from Audio tape

C
P
C/P
P/C

10-6-09 Reflection on back

1-5	P Give Q's student discuss in groups P/C	C groups discussing	CQ PQ > teacher	GQ > Kids teacher guides a group	C & P Q to student by Teacher
6-10	C "Good way to look @ it	P Directive "look @ #'s its part of the problem"	What is your labels writes on Bord to help associate where to put the #'s C	I want to see your Solutions. You didn't do that correct C's P > student's	did you cross X are they equal? P but a little in Your suppose to put label #'s C P
11-15 (P) (C)	You have a fraction in here C/P. Your not showing your How are you going to solve	Why did you ... C/P listening to student	Had to Show (P) we know Did take your Do you agree C	How are you going to figure that out students "C" K	Morgan your forgetting Writes on board the structure of prob w guiding Q's C/P P
16-20 P	what's this # here P/C remember I said you can use your Calc but had to Show	Explain ... C What do this is what we know P/C R	Cross P How are you going to do it on this one? C/P	Do your partners know Where do you get ... (C) D	Students C/P
21-25 P/C	R - So what have we been able to do? cross X, & why are we cross X?	What are we Setting up? C proportion right C/P R	we set up this "Alg Equation" C Making Connections Discusses Processes	Guess & Check figure out (This is better) Now what do I have to do? C/P	Why did I write this like this? ok so it means the same thing & Why are we doing that? Why did you have to ÷ by 9? C
26-30	Want to get "n" by itself R-C-P Couple Chpt. ago	were having trouble w/ problems & need a little bit more info #11 on 289 So	let's look @ what it should look like Showing problem	So why did she use this? Austin (R) these 2 have to be the same (P) problem	were not comparing the same thing (P) So what do we have to do to work this out? is that your final answer
31-35 C	= 3 N Does everyone agree w/ that. You look confused	I agree w # (C) but the decimal is in the wrong place So where should it be	C R - Explore using decimal C/P rules	744 = N ok put it over here Use logic Does it make sense that we can go 744 miles in	Explain what you said P/C Is that what you said 3600 seconds? C
36-40 C	student talks about how he set it up Why are you getting this same answer	Teacher C/P Shows mathematics of what both procedures were doing	one of the things you can do to check your answer P/C	Same idea to √ some rates transfer P/C	read it Set it up to what it says R - Sec hours P P/C
41-45	What do we know C as she R the P	I don't see 3½ in the problem where did you get that # C	Language What don't we know on the other side Showing thinking on Board	P Works out problem Have to know what they're asking	Challenge Problem figure out rates Pepsi problem
46-50	Q's Which one is the better buy? C your asking the right Q's R	PARTNERS talk Price per OZ? " " " Qt ? let's C/P go	Student C/P	partners Can use calc C I want you to know the process & not So much the calculation	Students C/P Stop You I've seen @ least 3
51-55 C/P	Kind of on the right track. Group 1 is doing Are you cross X? No were ... R Did you put a = sign here	Imply your trying to equal (C/P) What's another reason why C's are can't do that (C)	If we set up as a proportion @ least not the way it is there (C/P) Both refer to $1 C	How many OZ to a Qt. 2 1/4 = 2.1 Q So what do you think C/P What are we going to have to do.	Get same units So what do you want to get the units to? C/P R fl oz, Qt to lbs.
56-60	Are we going to put 32 over there? C/P Stuck .89 ÷ 20	Use Calc (P) not the big thinking .044 per oz. What will it cost per oz for this then you set	This is what got? R 1st of all it is not per oz that. Why is it not?	this one looks like this one looks like So where is her error (C/P) So, what do you think	does that make sense (C) C/P why did you ÷ by 32
60-65	Should we ÷ by 32? (C) Why are you saying (C) Let you ponder would look @ it tommorrow	were not quite ready! Go back to your desk.	work more w/ price per unit R P/C	One more place small amounts one more metacognition. Diag Pg 4	P/C Shows 34 different b/w processes (P) "precise" 2 ways to figure it out

Appendix G

Audit

Confidentiality Agreement for Auditor

I, the undersigned, hereby acknowledge that I will in no way disclose the identities of the subjects or convey known data from the study entitled: *Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics.* I will maintain participant confidentiality in all matters to which I have been given access relative to this study.

Signed: Date:

August 5, 2012

External Audit Attestation
By Kathy J. Fuchser, Ed.D.
Midland University

Michael J. Gay asked me to do an educational audit of his qualitative dissertation entitled *Excellent Teaching: A Collective Case Study of Outstanding Elementary Mathematics Teachers' Teaching of Mathematics*. He is a Ph.D. candidate in Educational Studies (Teaching, Curriculum and Learning). The audit was conducted August 1-August 5, 2012. The purpose of the audit was to determine the extent to which the results of the study are trustworthy.

Creswell (1998) indicated that the role of the auditor is to carefully examine "both the process and the product of the account, assessing their accuracy" and assuring that "the findings, interpretations, and conclusions are supported by the data" (p. 203).

The researcher delivered to me the following materials:

- A copy of the dissertation
- The confidentiality agreement for the auditor's signature.
- Copies of all letters of contact with IRB approval numbers.
- School permission forms with administrative signatures.
- Signed informed consent forms from teacher and administrator participants.
- Comprehensive dissertation planning notes.
- Research Log.
- 39 Audio Tapes of interviews and observations for all participants.
- One Research journals from each of three participants.
- 2 researcher research journal with handwritten analysis notes.
- CKT-M Workshop Notes.
- Codes and Cluster Code Cards.
- 2 Cross-Case reference charts.
- Dimension reference chart.
- Teacher selection guidelines and data.
- Field observation protocol, handwritten notes and data for individual participants.
- Interview protocol, handwritten notes and data for individual participants.
- Preliminary handwritten notes for each individual participant's interview transcripts.
- Preliminary handwritten themes from individual participant's interview transcripts.
- Original handwritten classroom observation notes for each individual participant.

The audit consisted of the following steps:

1. The Researcher delivered all materials to me.
2. I signed the auditor confidentiality agreement.
3. I catalogued and reviewed all materials submitted.
4. I thoroughly read the dissertation, paying particular attention to the purpose, grand tour question, and research questions identified.
5. I reviewed interview transcripts and classroom observations.
6. I reviewed the Codes and Clusters of Codes to confirm the reasonable deductions of the researcher.
7. I reviewed the Cross-Case Reference Charts against the Dimensions Chart.
8. I reviewed the process for identifying key issues for each participant.
9. I reviewed the processes and results of the Cross-Case analysis.
10. I considered how the noted literature provided support in identifying teaching dimensions.
11. I drew conclusions regarding the audit trail and trustworthiness of the findings.
12. I visited with the researcher regarding diligence in maintaining participant confidentiality and he made the appropriate changes based on the conversation.

It is the auditor's assessment that the researcher methodically designed and implemented the project over time and provided a clear trial for audit. The materials submitted were detailed and well-organized. Therefore, because the findings are reasonably grounded in data and analysis, the trustworthiness of this study can be established.

Attested to by Kathy J. Fuchser on August 5, 2012

Kathy J. Fuchser, Ed.D.
Associate Professor of Education
Midland University